Study of Navier-Stokes Equation Solution

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Abstract

Values of large dimensionless unknown functions (for example, a large Reynolds number) can be found out as solutions of non-linear partial differential equation. In this case these equations can be brought to some number of non-linear ordinary differential equations. Turbulent solutions corresponding to large values of unknown function are complex. Transition from real solution to complex turbulent solution is realized through infinity of the right parts of ordinary differential equation system to which Navier-Stokes equations are brought. Thus, real solution of Navier-Stokes equation for turbulent mode yields function going to infinity. At the same time, complex solution for the turbulent mode is finite. Fluid flow resistance coefficient is calculated for round pipeline with different pipeline walls roughness.

Problem Formulation

Let us consider Navier-Stokes problem and continuity equation for incompressible fluid. They are as following

$$\frac{\partial \mathbf{V}_k(t,\mathbf{r})}{\partial t} + \sum_{l=1}^{3} \mathbf{V}_l(t,\mathbf{r}) \frac{\partial \mathbf{V}_k(t,\mathbf{r})}{\partial x_l} = -\frac{\partial P(t,\mathbf{r})}{\rho \partial x_k} + \nu \Delta \mathbf{V}_k(t,\mathbf{r}), k = 1,...,3 \quad .$$
$$(\nabla, \mathbf{V}) = 0$$

Boundary conditions on the body boundary adjoining to fluid are $\mathbf{V}(t,\mathbf{r}) = 0, \mathbf{r} \in S$ where S is a body boundary. We will seek a solution in the form of series using Galerkin method (hereinafter $N \to \infty$)

$$\mathbf{V}_l(t,\mathbf{r}) = \sum_{n=1}^{N} \mathbf{x}_{nl}(t) \varphi_{nl}(\mathbf{r}), P(t,\mathbf{r}) = \sum_{n=1}^{N} y_n(t) \psi_n(\mathbf{r}) + \psi_0(\mathbf{r}); \quad ,$$
$$\mathbf{r} \in S \to \varphi_n(\mathbf{r}) = 0, \varphi_n(\mathbf{r}), \psi_n(\mathbf{r}) \in C^2$$

where space C^2 is twice continuously differentiable function, $\psi_0(\mathbf{r})$ is a defined external action which, in case of pipeline, is equal to $\psi_0(\mathbf{r}) = -(P - P_0) z / L + P$, where z - direction of the pipeline longitudinal, P, P_0- pressure at the beginning and the end of the pipeline, L - pipeline length.

Now we substitute these functions into the differential equation, multiply by $\psi_m(\mathbf{r})$ and integrate over the volume, then we obtain following differential equations system:

$$\begin{gathered}\frac{dx_m(t)}{dt} = \sum_{p,q=1}^{3N} F_{mpq} x_p(t) x_q(t) + \sum_{p=1}^{4N} G_{mp} x_p(t) + H_m; m = 1,...,3N \\ \sum_{p=1}^{3N} P_{mp} x_p(t) = 0, m = 1,..., N\end{gathered} . \qquad (A.1)$$

After we resolved the second equation (A.1), substituted $x_{n+2N}(t) = \sum_{m=1}^{2N} c_{nm} x_m(t), n = 1,..., N$ from the second equation (A.1) to the first one, we have

$$\begin{gathered}\frac{dx_m(t)}{dt} = \sum_{p,q=1}^{2N} F^1_{mpq} x_p(t) x_q(t) + (\sum_{p=1}^{2N} + \sum_{p=3N+1}^{4N}) G^1_{mp} x_p(t) + H_m; m = 1,...,2N \\ \sum_{p,q=1}^{2N} F^1_{mpq} x_p(t) x_q(t) + (\sum_{p=1}^{2N} + \sum_{p=3N+1}^{4N}) G^1_{mp} x_p(t) + H_m = 0, m = 2N+1,...,3N\end{gathered} . \qquad (A.2)$$

Defining $x_{n+3N}(t), n = 1,..., N$, corresponding to pressure change, from the second equation (A.2) and substituting found out value into the first equation (A.2), we have equations system

$$\frac{dx_m(t)}{dt} = \sum_{p,q=1}^{2N} F^2_{mpq} x_p(t) x_q(t) + \sum_{p=1}^{2N} G^2_{mp} x_p(t) + H^1_m; m = 1,...,2N . \qquad (A.3)$$

At that

$$x_{n+2N}(t) = \sum_{m=1}^{2N} c_{nm} x_m(t),$$

$$x_{n+3N}(t) = \sum_{m=1}^{N} c_{nm} H_m + \sum_{m=1}^{2N} b_{nm} x_m(t) + \sum_{p,q=1}^{2N} F^1_{mpq} x_p(t) x_q(t), n = 1,..., N$$

Where values

$$\mathbf{x}_{nl}(t) = x_{n+N(l-1)}(t), l = 1,...,3; y_n(t) = x_{n+3N}(t), n = 1,..., N .$$

are known and coefficients $F_{mpq}, G_{mp}, H_m, F^1_{mpq}, G^1_{mp}, H^1_m, F^2_{mpq}, G^2_{mp}, H^2_m, c_{nm}, b_{nm}$ are constants. This system of non-linear ordinary differential autonomous equations (A.3) is to be solved. Solution convergence issues will be discussed below in the text.

1. Finding of Solution of Ordinary Differential Equations in Complex Plane

Let us consider system of non-linear differential autonomous equations

$$\frac{dc_l}{dt} = Q_l(c_1,..., c_N), l = 1,..., N . \qquad (1.1)$$

Navier-Stokes equation system and continuity equation can be brought to system of non-linear differential equations:

$$\frac{dc_m(t)}{dt} = \sum_{p,q=1}^{N} F_{mpq} c_p(t) c_q(t) + \sum_{p=1}^{N} G_{mp} c_p(t) + H_m = Q(c_1,..., c_N); m = 1,..., N \quad (1.2)$$

where three-dimensional velocity is defined by formula $\mathbf{V}(t, x_1, x_2, x_3) = \sum_{n=1}^{N} \mathbf{c}_n(t) \varphi_n(x_1, x_2, x_3)$. At that, function $\varphi_n(x_1, x_2, x_3)$ is given in the form of sine or cosine. Then coefficients $c_n(t)$ for continuous function decrease not more rapidly than $1/n^2$ when index increases and series reduction is possible, i.e. instead of infinite number of terms, finite terms number is used. At the same time, the infinite number of terms forms convergent series.

It was found out that a set of $N+1$ coordinates for the system equilibrium position exists (1.2). Indeed, let us assume that we have found several equilibrium positions with coordinates $b_l^0, l=1,...,N$. Let us seek the solution in the form $b_l = b_l^0 + b_l^s$. For that we will substitute the solution into the right part of the differential equation (1.2) and will equate it to zero, then following equations system is obtained

$$\sum_{l=1}^{N} A_{kl}(b_1^s,...,b_N^s)b_l^s = 0.$$

For existence of non-zero solution of this differential equation, it is necessary that determinant is equal to zero: $|A_{kl}(b_1^s,...,b_N^s)|=0$. Given zero determinant, coefficients b_l^s from linear equation will be defined up to a multiplier. This multiplier will be defined from equality to zero of determinant of non-linear equation system. Thus, we have N unknown multipliers, which will be defined from determinant equality to zero. I.e. set of $N+1$ coordinates of the system equilibrium position exists.

Differential equation system (1.2) for non-multiple equilibrium positions can be expressed by means of $c_l = \sum_{k=1}^{N} g_{lk} x_k$ substitution. At that, the system (1.2) equilibrium positions $b_l^s, l=1,...,N; s=1,...,S$ will be transformed into equilibrium positions $a_l^s, l=1,...,N; s=1,...,S$ and eigen values and eigen vectors of the linearized system (1.2) will be defined as.

$$\begin{gathered}[\frac{\partial Q_k}{\partial c_m}(b_1^s,...,b_N^s) - \Lambda_\alpha^s \delta_{km}]g_{m\alpha}^s = 0 \\ |\frac{\partial Q_k}{\partial c_m}(b_1^s,...,b_N^s) - \Lambda_\alpha^s \delta_{km}| = 0\end{gathered}.$$

Equation system (1.2) will be

$$\frac{dx_n}{dt} = \Lambda_n^s(x_n - a_n^s) + \sum_{k=1}^{N} (x_n - a_n^k)^2 P_n^k(x_1,...,x_N) = F_n(x_1,...,x_N)\,. \qquad (1.3a)$$

Values a_l^s satisfy condition $F_k(a_1^s,...,a_N^s) = 0, k = 1,..., N; s = 1,..., S$.

Equation system (1.3a) can be written as

$$\frac{dx_l}{dt} = \exp[G_l(x_1,...,x_N)]\prod_{s=1}^{S} (x_l - a_l^s)\,, \qquad (1.3b)$$

where multiplier which can never be equal to 0 is used - $\exp[G_l(x_1,...,x_N)]$, and this multiplier is equal to $\exp[G_l(x_1,...,x_N)] = F_l(x_1,...,x_N)/\prod_{s=1}^{S} (x_l - a_l^s)$. After this multiplier is substituted to (1.3b) we obtain (1.3a). Now we will demonstrate that this multiplier can never be equal to 0. When $x_l \to a_l^\alpha, l = 1,..., N$ the limit of

$$\exp[G_l(a_1^\alpha,...,a_N^\alpha)] = \frac{\partial F_l(a_1^\alpha,...,a_N^\alpha)}{\partial x_l}/[(a_l^\alpha - a_l^1)...(a_l^\alpha - a_l^{\alpha-1})(a_l^\alpha - a_l^{\alpha+1})...(a_l^\alpha - a_l^S)] =$$
$$= \Lambda_l^\alpha/[(a_l^\alpha - a_l^1)...(a_l^\alpha - a_l^{\alpha-1})(a_l^\alpha - a_l^{\alpha+1})...(a_l^\alpha - a_l^S)]$$

is finite.

Here we canceled out a multiplier $x_l - a_l^\alpha$, as we consider only not coincident roots being coordinates of equilibrium position. So we showed that this multiplier does not equal to zero after infinite time.

Thus, the differential equation can be written as

$$\frac{dx_l}{dH_l(t,t_0)} = \prod_{s=1}^{S} (x_l - a_l^s)$$
$$\frac{dH_l(t,t_0)}{dt} = \exp\{G_l[x_1(H_l),...,x_N(H_l)]\}, l = 1,..., N \qquad , \qquad (1.4)$$

where $H_l(t,t_0)$ - function which tends to infinity when coordinates tend to equilibrium position. For real solutions, this function is monotonic. That is, we have obtained dependence of the solution on value $H_l(t,t_0)$, which is monotonic time-dependent function.

Lemma 2. Necessary and sufficient criterion for unknown function to tend to steady equilibrium position coordinates is $H_l(t,t_0) \to \infty$ when $t \to \infty$. At the same time, equilibrium position coordinates have to have a real part.

So, we have

$$\begin{gathered} \exp[G_l(x_1,...,x_N)] \to \exp[G_l(a_1^s,...,a_N^s)] = \\ = \Lambda_l^s / [(a_l^\alpha - a_l^1)...(a_l^\alpha - a_l^{\alpha-1})(a_l^\alpha - a_l^{\alpha+1})...(a_l^\alpha - a_l^S)]; \end{gathered} \tag{1.5}$$

at $t \to \infty$ and hence $H_l(t,t_0) \to \infty, l = 1,...,N$ as integral of constant. Inverse theorem is also valid, on condition $H_l(t,t_0) \to \infty, l = 1,...,N$, one of steady equilibrium positions is realized. This is a consequence of solution type; on condition $H_l(t,t_0) \to \infty, l = 1,...,N$, according to Lemma 4, negative real part of value λ_l^s exists in formula (1.6) and solution tends to equilibrium position coordinate a_l^s in formula (1.4). If equilibrium position coordinates have real parts, values λ_l^s have real part. At that $t \to \infty$.

Lemma 3. Solution of differential equation (1.1) is function $x_l(t)$ which satisfies to equation (1.6).

To obtain (1.6), let us divide equation (1.4) by product of multipliers $x_l - a_l^s$ and multiply (1.4) by $dH_l(t,t_0)$. Then we will decompose obtained fraction into sum of simple fractions and perform integration. The following equation is obtained

$$\sum_{s=1}^{S} \lambda_l^s [\ln(x_l - a_l^s) + 2\pi i n_s]|_{t_0}^{t} = H_l(t,t_0), l = 1,...,2N \ .$$

Here for the case of sound energy emission in interval $[t_0, t]$ different branches of logarithm are obtained.

After the expression exponentiated, we have (1.6)

$$\prod_{s=1}^{S} (x_l - a_l^s)^{\lambda_l^s} \exp(2\pi i \lambda_l^s \Delta n_s) / \prod_{s=1}^{S} (x_l^0 - a_l^s)^{\lambda_l^s} = \exp[H_l(t,t_0)]; \quad , \qquad (1.6)$$
$$\lambda_l^s = 1/[(a_l^s - a_l^1)...(a_l^s - a_l^{s-1})(a_l^s - a_l^{s+1})...(a_l^s - a_l^S)]$$

where all values of equilibrium position coordinates are not multiple and are not dependent on radiation process occurring in an interval $[t_0, t]$. In case of laminar real solution, radiation will not appear, and in case of turbulent solution, followed by radiation, there will be energy transition. Really, presence of radiation yields the complex solution which describes turbulent pulsing mode. At that, at solution transformation, turbulent mode is followed by sound noise. Exponential multiplier does not affect equilibrium position coordinates which define stationary solution. Existence of multiplier $\exp(2\pi i \lambda_l^s \Delta n_s)$ changes calculated main branch of solution for coordinate x_l, but will not affect equilibrium position coordinate.

Lemma 4. Sum of coefficients λ_l^s by index s is equal to zero, i.e. $\sum_{s=1}^{S} \lambda_l^s = 0$

In case if following fraction decomposed.

$$P(y) = \frac{Q_{S-1}(y)}{(y - a_l^1)...(y - a_l^{s-1})(y - a_l^{s+1})...(y - a_l^S)} \ .$$

where $Q_{S-1}(y)$ is $S-1$-ordered polynomial. Equation $\sum_{s=1}^{S} \lambda_l^s = 0$ will remain satisfied, $\lambda_l^s = \frac{Q_{s-1}(a_l^s)}{(a_l^s - a_l^1)...(a_l^s - a_l^{s-1})(a_l^s - a_l^{s+1})...(a_l^s - a_l^S)}$. Let us prove this. For this let us consider a sum

$$P(y) = \sum_{s=1}^{S} \frac{Q_{S-1}(a_l^s)(y - a_l^1)...(y - a_l^{s-1})(y - a_l^{s+1})...(y - a_l^S)}{(a_l^s - a_l^1)...(a_l^s - a_l^{s-1})(a_l^s - a_l^{s+1})...(a_l^s - a_l^S)}.$$

This sum is equal to $P(y) = Q_{S-1}(y)$. We write formula for polynomial equal to $Q_{S-1}(y)$, dividing the equation by product $(y - a_l^1)...(y - a_l^S)$ we obtain

$$\begin{aligned}
&\sum_{s=1}^{S} \frac{Q_{S-1}(a_l^s)}{(a_l^s - a_l^1)...(a_l^s - a_l^{s-1})(a_l^s - a_l^{s+1})...(a_l^s - a_l^S)(a_l^s - y)} + \\
&+ \frac{Q_{S-1}(y)}{(y - a_l^1)...(y - a_l^{s-1})(y - a_l^s)(y - a_l^{s+1})...(y - a_l^S)} = 0
\end{aligned}.$$

If suppose that $y = a_l^{S+1}$, equality $\sum_{s=1}^{S+1} \lambda_l^s = 0$ is satisfied when $S+1$ equilibrium position exists.

But to realize the solution, it is necessary to know equilibrium positions of this non-linear equations system. Besides, equilibrium positions can be multiple that changes the solution finding process, it becomes random or chaotic, but we are not going to consider this case. Nevertheless, it is possible to prove the following important theorem.

Theorem 1. Cauchy task is considered under arbitrary real initial conditions for system of non-linear ordinary differential equations (1.1). If system (1.1) has complex conjugate equilibrium positions with real parts then, for finite real argument t, Cauchy problem solution for the system (1.1), for real initial conditions, tends to infinity. Then this solution becomes a complex one, tending to equilibrium position in case when complex equilibrium position coordinates have real part. Here the right part of (1.1) is considered as being a regular function, real for real arguments. This function has finite number of non-multiple equilibrium positions.

Proving

If the system (1.2) is resolved at non-multiple equilibrium positions then, according to Lemma 3, we have

$$\begin{gathered}\{-2\lambda_{iml}^{s}\arctan[(x_l-a_l^s)/b_l^s]+\lambda_{rel}^{s}\ln[(x_l-a_l^s)^2+(b_l^s)^2]\}|_{t_0}^{t}+\\ +\sum_{k}\lambda_l^k\ln(x_l-c_l^k)|_{t_0}^{t}=H_l(t,t_0),\end{gathered}\quad ,\ (1.7)$$

where $a_l^s+ib_l^s$ selected complex equilibrium position, c_l^s other equilibrium positions. Coefficients λ_l^s satisfy condition $\sum_s \lambda_l^s=0$, according to Lemma 4. At that, in sum $\sum_{s=1}^{S}\lambda_l^s$ real part value λ_{rel}^s in case of complex solution λ_l^s presents twice as all values λ_l^s satisfy condition $\sum_s \lambda_l^s=0$, so we have formula $2\lambda_{rel}^s+\sum_k \lambda_l^k=0$.

Let us substantiate solution (1.7). For that we will modify two complex conjugate terms of the solution (for expression simplicity, index l is omitted)

$$\frac{\lambda_{re}^s+i\lambda_{im}^s}{x-a^s-ib^s}+\frac{\lambda_{re}^s-i\lambda_{im}^s}{x-a^s+ib^s}=\frac{2(x-a^s)\lambda_{re}^s-2b^s\lambda_{im}^s}{(x-a^s)^2+(b^s)^2},\qquad (1.8)$$

where $\lambda^s=\lambda_{re}^s+i\lambda_{im}^s$. After integration (1.8) over argument x, we obtain formula (1.7)

$$\lambda_{re}^s\ln[(x-a^s)^2+(b^s)^2]-2\lambda_{im}^s\arctan\frac{x-a^s}{b^s}.$$

The solution is

$$x_l(t) = a_l^s + b_l^s \tan D_l(t),$$

where

$$D_l(t) = \{\sum_k \lambda_l^k \ln(x_l - c_l^k)|_{t_0}^t + \lambda_{rel}^s \ln[(x_l - a_l^s)^2 + (b_l^s)^2]\}|_{t_0}^t - H_l(t,t_0)\}/2\lambda_{iml}^s =$$
$$= \{\sum_k \lambda_l^k + 2\lambda_{rel}^s + \sum_k \lambda_l^k \ln(1 - c_l^k / x_l) + \lambda_{rel}^s \ln[(1 - a_l^s / x_l)^2 + (b_l^s)^2 / x_l^2] -$$
$$- \sum_k \lambda_l^k \ln(x_l^0 - c_l^k) - \lambda_{rel}^s \ln[(x_l^0 - a_l^s)^2 + (b_l^s)^2] - H_l(t,t_0)\}/2\lambda_{iml}^s,$$
$$\sum_k \lambda_l^k + 2\lambda_{rel}^s = 0 .$$

At that, value of $\sum_k (\lambda_l^k c_l^k + 2\lambda_{rel}^s a_l^s)$ is real due to existence of complex conjugate equilibrium positions. Thus, for $|x_l| \to \infty$ and finite t, we have equation

$$x_l(t) = a_l^s + b_l^s \tan D_l(t). \quad (1.9)$$

Solution of this equation tends to infinity.

At that, solution of differential equation for rising $H_l(t,t_0)$, according to Lemma 2, can have complex roots

$$\sum_k \lambda_l^k [\ln(x_l - a_l^k) + 2\pi i \Delta n_k]|_{t_0}^t = H_l(t,t_0) \ .$$

At that, as equation $\sum_k \lambda_l^k = 0$ is satisfied according Lemma 4 and equilibrium positions have real parts, values with negative real part λ_l^k exist, so convergence to one of the equilibrium positions takes place. Real solution will tend to infinity at that existence, and uniqueness condition for Cauchy problem will be breached. According to Lemma 2, at $H_l(t,t_0)$ infinity, unknown function will tend to one of equilibrium positions. This position of equilibrium can not be real as the real solution is infinite.

This means that the solution will have a branching point and will tend to complex equilibrium position. That is, for equilibrium complex positions, finite complex solution is obtained at $H_I(t,t_0)$ change. Thus, in some point a complex solution will begin.

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End of the proof.

Now we will give an example describing this property of the differential equation, transition to the complex solution. So, for the differential equation, there can be a complex solution instead of infinite real one

$$\frac{dx}{dt} = 1 + x^2 .$$

And these equilibrium positions are purely imaginary, that is, the solution cannot tend to equilibrium position. And the real solution of this differential equation tends rapidly to infinity $x = \tan[t - t_0 + \arctan(x_0)]$.

Using an implicit solution finding scheme, we obtain the following equation

$$x = x_0 + (1 + x^2)\Delta t + 0(\Delta t)^2 .$$

Seeking solution in respect to unknown function x, we obtain the following implicit scheme

$$x = \frac{1 - \sqrt{1 - 4[x_0 + \Delta t + 0(\Delta t)^2]\Delta t}}{2\Delta t} .$$

This implicit scheme with constant step correctly describes solution tendency to infinity. At reduced calculation step, this scheme yields larger value of variable t, that is, it yields larger value of unknown function. That is, it correctly describes behavior of the differential equation solution up to infinity. When infinity is reached, under condition $x_0 > 1/(4\Delta t) - \Delta t - 0(\Delta t)^2$, the finite complex solution will be found. Numerical computation of this equation has validated this analysis of the solution obtained.

At that, the complex solution possesses new properties; it performs complex rotation around equilibrium position. At the same time the real solution tends to infinity, i.e. right part of the differential equation tends to infinity and existence and uniqueness condition for Cauchy problem are breached, so additional complex solution is arisen.

The solution for complex initial data is given by formula $x = \tan[t - t_0 + \arctan(x_0 + i\delta)]$ for any t. Thus, approximately we have

$$x(t) = -i\frac{\exp\{i[t - t_0 + \arctan(x_0 + i\delta)] - \exp\{-i[t - t_0 + \arctan(x_0 + i\delta)]\}}{\exp\{i[t - t_0 + \arctan(x_0 + i\delta)]\} + \exp\{-i[t - t_0 + \arctan(x_0 + i\delta)]\}} =$$

$$= i - 2i\exp\{2i[t - t_0 + \arctan(x_0 + i\delta)]\} + i\exp\{4i[t - t_0 + \arctan(x_0 + i\delta)]\} + ... =.$$

$$= i - 2i\exp[2i(t - t_0 + \alpha) - 2\beta] + i\exp[4i(t - t_0 + \alpha) - 4\beta] + ...$$

$$\arctan(x_0 + i\delta) = \alpha + i\beta$$

If we choose branch with positive β, we obtain converging series. At that, this fraction denominator never becomes zero.

That is, for real plane, finite solution does not exist. In complex plane, finite continuous solution exists in the case if equilibrium positions are not multiple.

But there is a question – what is the physical meaning of imaginary part of the solution?

2. Physical Meaning of Exact Complex Solution

So, for turbulent solution corresponding to complex equilibrium position coordinates, we have solution

$$x_l = \alpha_l^s + \beta_l^s \tan D(t).$$

The solution consists of step term in the form of delta-function and smooth part

$$x_l = \alpha_l^s + \beta_l^s \{\tan[D(t) - i0] - \tan[D(t) + i0]\}/2 + \beta_l^s \{\tan[D(t) - i0] + \tan[D(t) + i0]\}/2.$$

As, at averaging over period, tangents sum without taking into account step term is equal to zero, we will study the step term of the solution. At that, this solution has singularity when condition $D(t) = \pi(k + 1/2)$ is satisfied. Step term of the solution is

$$\begin{aligned} x_l = \alpha_l^s + \beta_l^s \sum_{k=-\infty}^{\infty} \{[\frac{1}{D(t) - \pi(k+1/2) - i0} - \frac{1}{D(t) - \pi(k+1/2) + i0}]/2 = \\ = \alpha_l^s + \pi i \beta_l^s \sum_{k=-\infty}^{\infty} \delta[D(t) - \pi(k+1/2)] + \\ + Vp[\frac{1}{D(t) - \pi(k+1/2)} - \frac{1}{D(t) - \pi(k+1/2)}]/2\} = \\ = \alpha_l^s + \pi i \beta_l^s \sum_{k=-\infty}^{\infty} \delta[D(t) - \pi(k+1/2)] \end{aligned}.$$

That is, imaginary medium pulse is originated. Imaginary velocity means flow rotation or oscillation; flow step is originated which will be destroyed in time $\Delta D(t) = \pi$ to originate repeatedly. Number of such steps is finite. But how to average this steps? You should pass to probabilistic interpretation of the description. That is, to average imaginary part over the period $D(h_l^k) - \pi(k+1/2)$. Then we have local complex average solution

$$< x_l > - \alpha_l^s = \pi i \beta_l^s < \delta(h_l) / \dot{D}_l(h_l) > / \pi = i\beta_l^s / \dot{D}_l(h_l^k), D_l(h_l^k) = \pi(k+1/2).$$

Continuous part of the solution has positive and negative parts which are compensated when averaged. To obtain a global average value it is necessary to average with respect to value k, so we have $< x_l > - \alpha_l^s = i\beta_l^s \sum_{k=-N}^{N} 1/\dot{D}_l(h_l^k)/2N$. We obtained complex velocity; imaginary part is defined up to multiplier. Real part of complex velocity corresponds to average value of velocity, and imaginary part is a mean square deviation. Simultaneously, there is a vortex motion consisting of positive and negative value of root from β_l^s.

Contribution of imaginary part to average value is equal to

$$< x_l >= \alpha_l^s \pm i\sqrt{\beta_l^s}\gamma_l \,; \gamma_l = 1/\dot{D}_l\,(h_l^k)$$

At that, module of average value, that defines real solution, is equal to $|< x_l >| = \sqrt{(\alpha_l^s)^2 + \beta_l^s \gamma_l^2}\,; \gamma_l = 1/\dot{D}_l\,(h_l^k)$, where equilibrium position coordinates and time are non-dimensional, then, as we calculate square root of imaginary part, we define branch $\beta_l^s > 0$. Thus, average single-valued solution is found.

This multiplier γ_l depends on the surface roughness and it is found from numerical experiment. As numerical experiment has shown, for round smooth pipeline the multiplier is equal to $\gamma_l = 1$. At that, the smooth pipeline has a constant, minimum, average module of roughness inclination tangent equal to $<|\tan\alpha|>= 1/R_{cr} = 1/2300$, that is associated with molecular roughness, see section 1.1. For this, one term of series which determines flow velocity is used. We calculate this value for one term of the series for smooth surface. The solution is $x_l(t) = \alpha_l^s + \beta_l^s \tan(h_l)$ where

$$D_l(h_l^k) = h_l^k = \pi(k + 1/2); \dot{D}_l(h_l^k) = 1, \gamma_l =< 1/\dot{D}_l(h_l^k) >= 1.$$

This value exactly corresponds to experimental formula for round cross section pipeline if the solution roughness is taken into account. At that, to take roughness into account for internal problem, h_l^k is multiplied by $(\frac{1 + kR_{cr}/l}{2})^\sigma$, here k/l is a constant average tangent of flowing surface inclination. From this we obtain $\gamma_l = (\frac{2}{1 + kR_{cr}/l})^\sigma$, see section 1.1. At constant average roughness height, the coefficient γ_l is not a constant as k/l value is determined by other formula depending on dimensionless pressure, see section 1.1.

Appendix

1. Laminar Solution of Navier-Stokes Equation

Value of round pipeline resistance coefficient for arbitrary Reynolds number and roughness degree are known only from experiment. It is proposed, using complex solution, to obtain a solution of Navier-Stokes equations and based on the qualitative reasons to define roughness influence on the solution of Navier-Stokes equation. It was possible to draw classical Nikuradze curves for round pipeline resistance coefficient versus Reynolds number and roughness degree with an accuracy of 10%.

Introduction

The problem of turbulent fluid motion description has not been solved yet. It creates difficulties when oil, gas pipelines design calculation is performed. Besides, there are no theoretical methods for description of bodies motion in turbulent environment. These methods would be necessary for description of motion of aircrafts, submarines or above-water ships in the turbulent mode. Without simulation of bodies motion in wind tunnels or water basins, design of the bodies moving in the viscous environment is impossible.

There are approximate formulas for pipeline resistance coefficient at some ranges of Reynolds numbers, see [1], [2]. But they are empirical approximate formulas and they are applicable only for particular Reynolds number ranges.

Classical experimental Nikuradze curves of round pipeline resistance coefficient versus Reynolds number and roughness degree are well known. Approximation of convective term reducing Navier-Stokes problem to linear one with effective turbulent viscosity is applied. But such transformation distorts solution of Navier-Stokes equations and for matching to experiment the turbulent viscosity coefficient can have any value, up to negative. Galerkin method which brings hydrodynamic problem solution to system of non-linear ordinary differential equations is applied. But in the case of turbulent mode, this non-linear equation system has complex equilibrium positions, i.e. the solution is complex. Indeed, hydrodynamics equation system, in turbulent mode, in real plane does not have any solutions, the equation solution tends for infinity, see [3] and main part of the paper. But complex solution is finite. About physical meaning of the complex solution and oscillatory behavior of its imaginary part see [4],[5] or Section 2 of this paper. Thus, it is necessary to solve hydrodynamic problems for turbulent mode in complex plane. At that, the turbulent solution is not single-valued, there are finite number of the solution branches.

1.1. Calculation of Round Section Pipeline Resistance Coefficient for Incompressible Fluid

This algorithm has been used for calculation of resistance coefficient for pipeline with round cross section. The algorithm is described in article [6] in English. We will seek the solution of problem for cross section round pipeline in form $V_z = V_0(t)[1 - r^2 / a^2(z)]$, in cylindrical system of coordinates. As external factor acts only along longitudinal axis $P(z) = P_2 + \frac{P_1 - P_2}{L} z$, where P_2, P_1 - pressure in initial and final part of the pipeline, L - length of the pipeline, radial and angular velocities components are neglected. External action is equal to $h_z = \frac{P_1 - P_2}{L}$. According to formula (1.2.2), the pressure gradient is equal to $\frac{\partial P}{\partial z} = \frac{P_1 - P_2}{L}$. So we have the equation

$$\frac{\partial V_z}{\partial t} + V_z \frac{\partial V_z}{\partial z} = -\frac{P_1 - P_2}{L} + \nu \Delta V_z$$

Substituting velocity value we obtain

$$\frac{\partial V_0}{\partial t}(1-r^2/a^2)+2V_0^2(1-r^2/a^2)\frac{r^2}{a^3}\frac{da}{dz}=-\frac{P_1-P_2}{\rho L}-\nu\frac{4V_0}{a^2} \quad (1.1.1a)$$

Multiplying this equation by radius and performing integration over radius, as we use cylindrical coordinate system, we have

$$\frac{\partial V_0}{\partial t}a^2/6+\frac{(P_1-P_2)a^2}{2\rho L}+2\nu V_0=-V_0^2\frac{ada}{6dz}.$$

To obtain finite number of solutions we will multiply equation (1.1.1a) by function $r(1-r^2/a^2)^n$ and integrate over volume. Then we will obtain finite number of turbulent solutions both for smooth and rough surfaces. Stationary laminar solution satisfying condition $da/dz=0$ is single-valued as in the equation (1.1.1a) the laminar solution is identical for different values of $r(1-r^2/a^2)^n$. At the same time, likewise Schrödinger equation, finite number of turbulent solutions is found, each has its own energy. At transition from one state to another, discrete energy is radiated. The own energy minimum value defines the solution choice.

After calculating a module of the right part and averaging module of deviation angle tangent, we obtain

$$\frac{\partial V_0}{\partial t}a^2/6+\frac{(P_1-P_2)a^2}{2\rho L}+2\nu V_0=V_0^2\frac{a<|da/dz|>}{6}=V_0^2\frac{2ak}{l} \quad (1.1.1b)$$

It will be seen that when minus sign is chosen for value of average module of deviation angle tangent $<|da/dz|>$, roughness presence increases flow velocity as the full derivative $\frac{dV_0}{dt}=\frac{\partial V_0}{\partial t}-V_0^2\frac{<|da/dz|>}{a}$ increases and this is not correct, flow velocity has to decrease due to roughness presence.

When turbulent viscosity is taken into account, negative value of average velocity associated with process velocity correlation function $-\rho<u_l' \, u_\alpha'>=\rho K\frac{\partial<u_l'>}{\partial x_\alpha}$, see [1], is used and this leads to plus sign for average module of roughness inclination tangent. The movement equation taking into account disturbances is

$$\frac{\partial < \rho u_l >}{\partial t} + \frac{\partial}{\partial x_\alpha}(\rho < u_l >< u_\alpha > + \rho < u'_l u'_\alpha >) = -\frac{\partial < p >}{\partial x_l} + \rho \nu < \Delta u_l >$$

That is, convection term should be taken with minus, at right part of (1.1.1b) should be taken with plus.

Besides, it is necessary to choose plus for average module of roughness inclination tangent to obtain complex turbulent solution. Otherwise, solution describing pulse turbulent mode will not be steady.

Changing pipeline radius to diameter and dividing by value $v^2 k/(dl)$, we obtain

$$\frac{dR_0}{d\tau} = R_0^2 - 2R_0 R_{cr} + \frac{T}{8}; T = \frac{(P_2 - P_1)d^3 R_{cr}}{\rho v^2 L} \quad . (1.1.2)$$

$$\tau = 24t \cdot v/(R_{cr} d^2), R_0 = V_0 d/v, 1/R_{cr} = <|da/dz|>/12 = k/l = <|\tan\varphi|>$$

If you use another branch of root mean square unsteady solution and following equation will be obtained

$$\frac{dR_0}{d\tau} = -R_0^2 - 2R_0 R_{cr} + \frac{T}{8}. \qquad (1.1.3)$$

Thus, steady solution for large difference in pressure is

$$R_0 = -R_{cr} + \sqrt{R_{cr}^2 + T/8}.$$

Laminar solutions of these two equations at small pressure difference are the same. For turbulent mode with big pressure the solution has linear dependence of Reynolds number versus pressure square root. At small pressure increase, Reynolds number also grows and, as it follows from (1.1.3), pressure is increased. So the solution is not steady. In case of the complex solution it is equal to

$$R_0 = R_{cr} - i\sqrt{T/8 - R_{cr}^2}.$$

At that, when pressure increases, imaginary part of velocity increases too and this does not lead to increase of real pressure, the real pressure keeps the value unchanged.

If micro roughness $<|\tan\varphi|>$ is distributed all over the pipeline surface, it is also present on macro roughness and defines critical Reynolds number and resistance coefficient at Reynolds number 2300. Micro roughness has the molecular nature, it is defined by average atom size equal to average geometrical difference between the nuclear size r_A and size of Bohr orbit $\sigma = \sqrt{r_A a_0}$ when the distance between atoms $a = 3.043A$ is equal to some value determined by properties of pipeline boundary, iron, titanium and carbon. Distance between iron atoms is $a_{Fe} = 2.87A$, between titanium atoms - $a_{Ti} = 3.46A$, between carbon atoms - $a_C = 3.567A$, see [7]. At the same time, the absolute value of tangent of micro roughness height inclination for metal surface of the pipeline is determined by formula $h(z) = <|\tan\varphi|> = \sum_{n=-N}^{N} \exp[-(z-na)^2/2\sigma^2]/(2N\sqrt{2\pi})$. The average tangent of inclination is equal to

$$\frac{1}{R_{cr}} = \int_{-\infty}^{\infty} h(z)\frac{dz}{2Na} = \frac{\int_{-\infty}^{\infty} \exp[-(z-na)^2/2\sigma^2]dz}{2\sqrt{2\pi}a} = \frac{\sigma}{2a} = .$$

$$= \frac{1}{2\cdot 3.043}\sqrt{\frac{r_A}{a_0}} = \frac{1}{2\cdot 3.043}\sqrt{\frac{1.4\cdot 10^{-13}}{0.5\cdot 10^{-8}}} = \frac{1}{1150}$$

In this paper, critical Reynolds number was calculated with respect to radius. Critical Reynolds number with respect to diameter is equal to $R_{cr} = 2300$. But why critical Reynolds number for the sphere is equal to $3\cdot 10^5$? This is due to different definition of critical Reynolds number. This value is equal to $\frac{1}{R_{cr}} = \frac{da}{ds} = \frac{dl_{eff}}{ds}\cdot\frac{a}{l_{eff}} = \frac{1}{2300}\cdot\frac{a}{l_{eff}}$, where l_{eff} - effective hydrodynamic size of the body, including medium, a - true geometrical body size, and $\frac{dl_{eff}}{ds} = |\tan\varphi| = \frac{1}{2300}$ - molecular tangent of roughness inclination. And the ratio

$\frac{a}{l_{eff}}$ can be equal to $\frac{a}{l_{eff}} = 0.01$.

Critical Reynolds number is equal to $R_{cr} = 2300$. Macro-roughness elements $<|da/dz|>$ are rarer and this causes increase of resistance coefficient at Reynolds numbers which is 12 or more times more.

So we obtained a stationary criterion for Navier-Stokes equations taking into account one term of the solution series for one-dimensional case:

$$R_0^2 - 2R_0 R_{cr} + T/8 = 0$$

For one-dimensional case, on condition of pipeline cross section area constancy, the continuity equation is the same. Laminar solution of this equation is

$$R_0 = R_{cr} - \sqrt{R_{cr}^2 - T/8} = [R_{cr}/\sqrt{T} - \sqrt{R_{cr}^2/T - 1/8}]\sqrt{T}\ .$$

For external pressure equal to $T = 8R_{cr}^2$, a complex solution and turbulent mode take place as Reynolds number from this point is equal to critical value. From experiment and calculation we have critical Reynolds number for round pipeline $R_{cr} = \frac{l}{k} = \frac{1}{<|\tan\varphi|>} = 2300$. The pipeline resistance coefficient for round cross section pipe is determined by formula (we substituted to the formula the pressure difference expressed through dimensionless pressure)

$$\lambda = \frac{2\Delta P_L d}{\rho V_a^2 L} = \frac{2T\nu^2 k}{V_a^2 d^2 l} = \frac{2T}{R_{cr}\,|R_a^2|},$$

The average velocity used for Reynolds number is equal to

$$V_a = \int_0^a rV_0(1 - \frac{r^2}{a^2})dr / \int_0^a rdr = V_0/2, R_a = \frac{V_a d}{\nu} = \frac{R_0}{2}.$$

The pipeline resistance coefficient λ_{lam} asymptotic for laminar mode in round cross section pipeline is calculated truly.

$$R_a = R_0/2 = (R_{cr} - \sqrt{R_{cr}^2 - T/8})/2 \cong \frac{T}{32R_{cr}}, \frac{T}{8R_{cr}^2} << 1, \lambda_{lam} = \frac{2T}{R_{cr}\,|R_a^2|} = \frac{64}{|R_a|}$$

Asymptotic behavior of the pipeline resistance coefficient is obtained for small Reynolds numbers when the convective term is small.

In case of large pressure difference, we have a complex turbulent solution $R_0 = R_{cr} - i\sqrt{T/8 - R_{cr}^2} = (R_{cr}/\sqrt{T} - i\sqrt{1/8 - R_{cr}^2/T})\sqrt{T}$. Computing more precisely, contribution of rotary imaginary part to forward velocity of flow movement corresponds to square root of imaginary part according to formula (1.1.4)

$$\frac{R_0}{\sqrt{T}} = \frac{R_{cr}}{\sqrt{T}} - i\sqrt[4]{\frac{1}{8} - \frac{R_{cr}^2}{T}}\sqrt{\beta} = \sqrt{\frac{R_{cr}^2}{T} + \sqrt{\frac{1}{8} - \frac{R_{cr}^2}{T}}\beta}\exp(i\psi)$$

$$\beta = \{\alpha/[k(T,\xi_0)R_{cr}/l(T,\xi_0)+1]\}^{\sigma}, \qquad (1.1.4)$$

$$\sigma = 0.25 \cdot 3/2 = 3/8$$

and it is necessary to use value of ratio of Reynolds number to square root of dimensionless pressure as value of order 1 in the turbulent mode. At infinite pressure, Reynolds number for the flow is proportional $R \sim \sqrt{T} \sim d_{eff}^{3/2}$. At that, the smoothest surface is the surface with average module of inclination tangent equal to inverse value of critical Reynolds number. For solution in the form of series, another value of α will be calculated. This value is defined from identical values of resistance coefficients at large Reynolds numbers and molecular roughness. The smoothest surface corresponds to average module of tangent of inclination equal to the inverse value of critical Reynolds number as the smallest modules of tangent of inclination correspond to molecular type of roughness. At that, effective diameter is less than true diameter. The average module of tangent of inclination angle can not be less than molecular roughness and its minimum value is equal to $<|\tan\varphi|> = 1/R_{cr}$. That is, 1 is the maximum value of ratio of effective diameter to true diameter because $\alpha = 2$. For external problem, effective diameter will increase, and the coefficient will be determined by formula $\beta = \{[k(T,\xi_0)R_{cr}/l(T,\xi_0)+1]/2\}^{\sigma}$.

Coefficient β is proportional to

$$\sqrt{T} \sim \beta = < d_{eff}^{3/2} > / d^{3/2} = \{2/[k(T,\xi_0)R_{cr}/l(T,\xi_0)+1]\}^{\sigma}, \sigma = \frac{1}{4}\frac{3}{2} = \frac{3}{8}.$$

At zero macro roughness, effective diameter is equal to 1, that is, when roughness is increased, effective diameter decreases. Value $d_{eff}/d = [2/(kR_{cr}/l+1)]^{1/4}$ was obtained from numerical experiment that corresponds to fourth root of mean square deviation. At zero macro roughness, micro roughness presents. And ratio of tangent of macro inclination roughness to micro roughness is more than $k/(l <|\tan\alpha|>) = 1$.

At $l/k = 30$, we have value of effective pipeline diameter $d_{eff}/d = [2/(2300/30+1)]^{1/4} = 0.38$.

At the same time, diameter is changed only for coefficient of pulsing part of the solution, i.e. for imaginary part from where the multiplier $\beta = \{2/[k(T,\xi_0)R_{cr}/l(T,\xi_0)+1]\}^{\sigma}$ originates as the imaginary term is proportional to $\sqrt{T} \sim d_{eff}^{3/2}$ which is averaged. At that, value $\sqrt[4]{1/8 - R_{cr}^2/T}$ corresponds to fourth root of mean square deviation.

Here, influence of walls roughness in turbulent flow on imaginary part of Reynolds number of the flow is taken into account. To obtain curves with constant roughness height, it is necessary to enter effective average module of tangent of roughness inclination angle. The effective average module of tangent of roughness inclination angle has to depend on external pressure $\dfrac{k(T,\xi_0)}{l(T,\xi_0)}$.

And in points of infinite Reynolds numbers or dimensionless pressure, we have the roughness corresponding to constant roughness height $\dfrac{k(\infty,\xi_0)}{l(\infty,\xi_0)} = \dfrac{k}{r_0} = \dfrac{1}{\xi_0}$, where k - mean square root of the roughness height, r_0 - radius of round cross section of the pipeline.

The formula is chosen in such a way that it defines correctly dependence of Reynolds number versus external pressure and pipeline resistance coefficient at infinite Reynolds numbers and external pressure

$\operatorname{Im} R_0 = -i\sqrt[4]{1/8}\{2/[k(\infty,\xi_0)R_{cr}/l(\infty,\xi_0)+1]\}^{\sigma}\sqrt{T}$ at resistance coefficient equal to $\lambda = \dfrac{16\sqrt{2}}{R_{cr}[2/(R_{cr}/\omega\xi_0+1)]^{2\sigma}}$.

When average module of tangent of roughness inclination angle $\dfrac{k}{l}$ is constant but roughness height k is varying we obtain a curve which differs from Nikuradze curve.

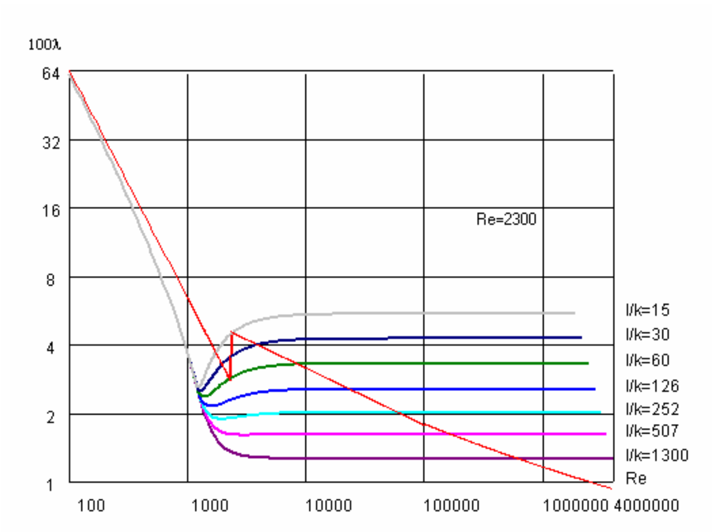

Fig.1 Curve of round pipeline resistance coefficient versus Reynolds number for different mean square root tangent of roughness inclination angle

But Nikuradze formula is obtained for constant ratio of pipeline radius r_0 to average roughness height k. The formula (1.1.4) contains effective average module of tangent of roughness inclination angle expressed through ratio of pipeline radius to average roughness height using dimensionless pressure

$$\frac{l(T,\xi_0)}{\delta(T,\xi_0)} = \{\exp[-|\sqrt{T}-\sqrt{T_{cr}}|/|\alpha(\xi_0)|] + \xi_0[1-\exp(-|\sqrt{T}-\sqrt{T_{cr}}|/|\alpha(\xi_0)|)]\}\times$$
$$\times\{1+0.4\exp\{-[\sqrt{T}-\sqrt{T_{cr}}\beta(\xi_0)]/\gamma(\xi_0)\}\},\xi_0 = r_0/k,$$

Value $T_{cr} = 8R_{cr}^2$. Influence of effective average module of tangent of roughness inclination on flow property depends on Reynolds number or pressure difference.

Empirical formula for finding of coefficients $\alpha(\xi_0), \beta(\xi_0), \gamma(\xi_0)$ is following

$$\alpha(\xi_0) = R_{cr}\frac{\xi_0}{1.5}, \qquad \beta(\xi_0) = \frac{\xi_0}{4}, \qquad \gamma(\xi_0) = R_{cr}\xi_0^{1.5}/4$$

At the same time, at the beginning of formation of the complex solution imaginary part $T = T_{cr} = 8R_{cr}^2$, or at the beginning of turbulent solution, roughness inclination tangent is equal to approximately 1, and curves for different roughness inclination tangents coincide.

At that, flow resistance coefficient for round pipeline is determined by formula $\lambda = \dfrac{2T}{R_{cr}\,|R_a|^2}$, Reynolds number calculated based on the average velocity of flow movement is equal to $R_a = R_0/2$. Resistance coefficient at infinite pressure is proportional to $\lambda = \dfrac{16\sqrt{2}}{R_{cr}[2/(R_{cr}/\xi_0 + 1)]^{2\sigma}}$. Here we demonstrate curves for solution obtained using one term of the series.

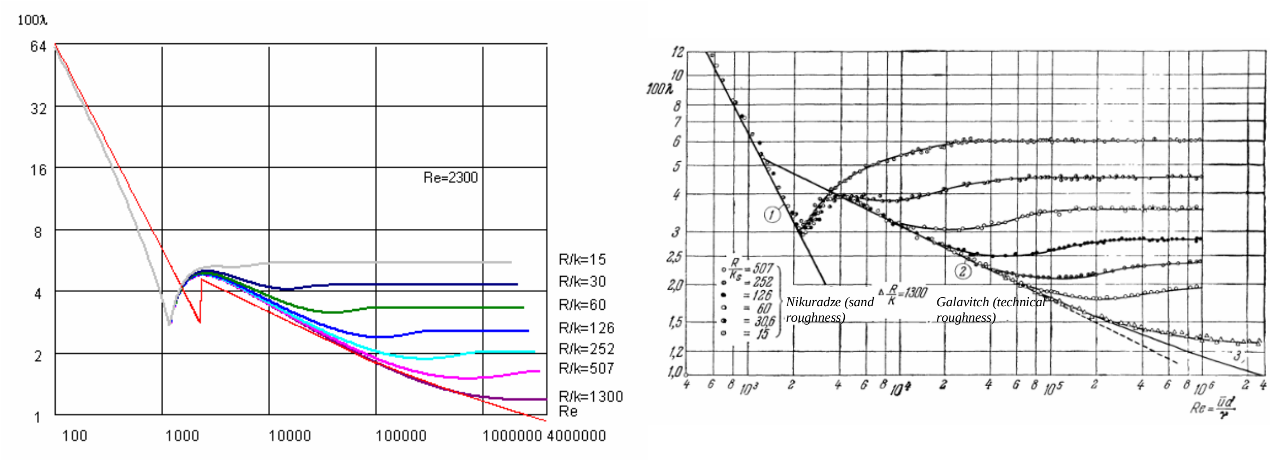

Fig.2 Calculated and measured dependence of round pipeline resistance coefficient versus Reynolds number for different roughness

To compare theoretical and experimental curves of resistance coefficient dependence versus flow Reynolds number, experimental curve by Nikuradze is given in Fig. 2, on the right. Error of the theoretical curve relative to experimental one is about 10%. But for laminar mode two solutions (1.1.2) and (1.1.3) are possible. Averaged solution will yield zero convection term and dependence $\lambda = 64/R_a$ that is not taken into account at the computation. In the theoretical curve convective term is taken into account which became equal to zero after averaging in laminar mode.

This curve was calculated for constant flow temperature over the flow cross section therefore in case of weak dependence of kinematic viscosity on temperature the formula will not change. For turbulent mode, it is necessary to substitute into the formula normalized pressure and ratio of pipeline radius to roughness height

$$| R_0 | = \sqrt{R_{cr}^2 + \sqrt{T^2/8 - TR_{cr}^2}\beta}$$

$$\beta = \{2/[k(T,\xi_0)R_{cr}/l(T,\xi_0)+1]\}^{\sigma}$$

$$\frac{l(T,\xi_0)}{k(T,\xi_0)} = \{\exp[-|\sqrt{T}-\sqrt{T_{cr}}|/\alpha(\xi_0)] + \xi_0[1-\exp(-|\sqrt{T}-\sqrt{T_{cr}}|/\alpha(\xi_0)]\}\times$$

$$\times\{1+0.4\exp\{-[\sqrt{T}-\sqrt{T_{cr}}\beta(\xi_0)]/\gamma(\xi_0)\}, \xi_0 = \frac{r_0}{\delta_0}$$

And the formula is constructed so that $\dfrac{l(\infty,\xi_0)}{k(\infty,\xi_0)} = \xi_0$. In case of the laminar mode there is a simple formula for Reynolds number: $R_0 = R_{cr} - \sqrt{R_{cr}^2 - T/8}$.

1.2. Algorithm for Solution of Internal Hydrodynamic Problem for Arbitrary Flow Geometry

Navier-Stokes equations in Cartesian coordinates is

$$\frac{\partial V_i}{\partial t} + \sum_{k=1}^{3} V_k \frac{\partial V_i}{\partial x^k} = -\frac{\partial P}{\rho \partial x^i} + \nu \Delta V_i . \ (1.2.1)$$

We will solve a three-dimensional laminar stationary problem without convective term for defined external action g_l

$$\frac{\partial P}{\rho \partial x_i} = \nu \Delta V_i .$$

Let us transform this equations to dimensionless form by dividing it by ν^2/d^3, as a result we obtain dimensionless equation

$$\frac{\partial p}{\partial y_i} = \Delta R_i,$$

$$R_s = V_s d / v, p = \frac{Pd^2}{\rho v^2}, y_s = s / d, h_s = g_s d^2 / v^2$$

Following function is a solution of this problem

$$R_s(y_1, y_2, y_3) = -\int\limits_V \frac{1}{4\pi |\mathbf{y}-\mathbf{z}|} \frac{\partial p}{\partial z_s} dz_1 dz_2 dz_3.$$

We will seek solution of continuity equation for external action, where r_i - response to external action

$$\frac{\partial R_i - r_i}{\partial x^i} = \int\limits_V \frac{y_s - z_s}{4\pi |\mathbf{y}-\mathbf{z}|^2} (\frac{\partial p}{\partial z_s} - h_s) dz_1 dz_2 dz_3 = 0 \quad (1.2.2)$$

From this we obtain equation for finding of flow pressure

$$\int\limits_V \frac{y_s - z_s}{4\pi |\mathbf{y}-\mathbf{z}|^2} \frac{\partial p}{\partial z_s} dz_1 dz_2 dz_3 = \int\limits_V \frac{y_s - z_s}{4\pi |\mathbf{y}-\mathbf{z}|^2} h_s dz_1 dz_2 dz_3.$$

We will seek the pressure value in the form $p = \sum\limits_{n=0}^{N} a_n \varphi_n(z_1, z_2, z_3)$. Then we will substitute pressure into expression under the integral sign, multiply by $\varphi_m(y_1, y_2, y_3)$, and perform integration over the volume, then we obtain a system of linear equation

$$b_m = A_{mn} a_n.$$

Expressions for coefficients are

$$A_{mn} = \int\limits_V \int\limits_V \varphi_m(y_1, y_2, y_3) \frac{y_s - z_s}{4\pi |\mathbf{y}-\mathbf{z}|^2} \frac{\partial \varphi_n(z_1, z_2, z_3)}{\partial z_s} dz_1 dz_2 dz_3 dy_1 dy_2 dy_3$$

$$b_m = \int\limits_V \int\limits_V \varphi_m(y_1, y_2, y_3) \frac{y_s - z_s}{4\pi |\mathbf{y}-\mathbf{z}|^2} h_s(z_1, z_2, z_3) dz_1 dz_2 dz_3 dy_1 dy_2 dy_3$$

where $h_l(y_1, y_2, y_3)$ is defined by external action. Let us transform Navier-Stokes equations to dimensionless form by dividing it by v^2 / d^3, and we have dimensionless equation

$$\frac{\partial \Re_l}{\partial \tau} + \sum_{k=1}^{3} \Re_k \frac{\partial \Re_l}{\partial y_k} = -\frac{\partial p}{\partial y_l} + \Delta \Re_l$$

$$\Re_l = \frac{V_l d}{v}, y_l = x_l / d, \tau = tv / d^2, p = \frac{Pd^2}{\rho v^2}, h_l = g_l \frac{d^2}{v^2} = \frac{\partial p}{\partial y_l} .$$

Then we multiply Navier-Stokes equations by area of flow tube cross section, write the equations along laminar solution and enter flow tube with constant flow, see [8]. $\Gamma_s = \int\limits_{S_s} \Re_s ds_s / d^2$. In the convection term and in pressure gradient, we enter the derivative in the direction corresponding to the direction of flow lines in laminar solution. When substituting of the solution into equation

$$\Gamma_s = \alpha_s(\tau) R_s[y_1(\alpha,\beta), y_2(\alpha,\beta), y_3(\alpha,\beta)] \quad (1.2.3)$$

where S_s - flow tube cross section in laminar mode, expression $R_s[y_1(\alpha,\beta), y_2(\alpha,\beta), y_3(\alpha,\beta)]$ is a stationary solution of Navier-Stokes equations without convection term which is equal to zero for flow tube as it does not depend on longitudinal coordinate.

We built these flow tubes for any external action which affects pressure difference. Further we consider roughness and under certain conditions obtain complex turbulent solution which is associated with influence of quadratic convection term with small multiplier, taking into account roughness, which yields complex solution at large pressure difference. At the same time, we reject real solution which was obtained for another sign of the module of average deviation, as it does not define fluctuating, turbulent solution. And imaginary part of the solution defines the solution pulsations.

If another sign of square root is chosen and correlation function of the process $< u_l' \, u_k' >$, where u_k' is a velocity deviation from its average value, is taken into account, turbulent viscosity becomes negative.

Let us substitute the solution (1.2.3) into Navier-Stokes equation, integrate it over flow pipe and divide by pipeline cross-sectional area. Then the convective term will be equal to

$$\sum_{k=1}^{3} \Re_k \frac{\partial \Re_l}{\partial y_k} = -\alpha_s^2(\tau)\frac{da}{ds}\int\limits_{S_s} R_s^2[y_1(\alpha,\beta), y_2(\alpha,\beta), y_3(\alpha,\beta)]d\alpha d\beta$$

Taking roughness into account results in dependence of the pipeline radius $a_0(s)$ on macro-roughness. Further we will extract the term da_0/ds associated with roughness and will find average value of its module. At the same time we will make averaging of the equation with respect to s. It can be found out that convection term in laminar mode for smooth surface is equal to zero, and roughness has to be taken into account for non-zero value. So, we have the equation

$$\frac{\partial \alpha_s}{\partial \tau}\int\limits_{S_s} R_s d\alpha d\beta = \alpha_s^2 < \frac{da}{ds} > \int\limits_{S_s} R_s \frac{\partial R_s}{\partial a} d\alpha d\beta - \frac{\partial \int\limits_{S_s} p d\alpha d\beta}{\partial s} + \alpha_s \int\limits_{S_s} \Delta R_s d\alpha d\beta .$$

To take into account roughness of pipeline surface and obtain turbulent solution, it is necessary to consider the average module of tangent of roughness inclination angle. Then this convective term will have a small multiplier, and the convection term will be non-zero. This term is proportional to average value of tangent of module inclination at roughness $<|\frac{da_0}{ds} - <\frac{da_0}{ds}>|>$. At the same time, there is a term depending on variable pipeline cross section area $\frac{d<a_0>}{ds}$. And flow lines of complex turbulent solution corresponding to flow lines of laminar solution will remain the same but there will be a solution pulsing around laminar flow lines. At that, the pulsations are defined by imaginary part of velocity, and the imaginary part of the solution, equal to a constant, means pulsations with amplitude equal to imaginary part of velocity.

Now, we will substitute the solution (1.2.3) into Navier-Stokes equations and will integrate along flow tubes, will multiply by R_{cr} in domain where this value meets a condition $1/R_{cr} = <|\tan\alpha|>$ and where $<|\tan\alpha|>$- average module of inclination tangent for not removable micro roughness with envelope forming macro-roughness, and we will obtain the following equation

$$R_{cr}\frac{d\alpha_s(\tau)}{d\tau} = F_s\alpha_s^2 - 2R_{cr}\alpha_s G_s + H_s$$

$$F_s = (R_{cr} < \frac{da_0}{ds} > +1)\int_{S_s} R_s[y_1(\alpha,\beta), y_2(\alpha,\beta), y_3(\alpha,\beta)]\frac{\partial R_s}{\partial a}d\alpha d\beta\,.$$

$$G_s = -\int_{S_s} \Delta R_s[y_1(\alpha,\beta), y_2(\alpha,\beta), y_3(\alpha,\beta)]d\alpha d\beta/2 > 0$$

$$H_s = -\int_{S_s} \frac{\partial p[y_1(\alpha,\beta,s), y_2(\alpha,\beta,s), y_3(\alpha,\beta,s)]}{\partial s}R_{cr}d\alpha d\beta ds > 0$$

where $R_s(y_1, y_2, y_3), p(y_1, y_2, y_3)$ are determined from laminar solution and continuity equation, and function of external action $h_l(y_1, y_2, y_3)$ is defined. So it was found out that micro roughness located along all length of the pipeline defines critical Reynolds number. This micro roughness is less than macro-roughness which affects resistance coefficient at large Reynolds numbers. But as Reynolds number depends on pipeline geometry through its diameter, then critical Reynolds number is inversely proportional to the average module of tangent of micro roughness inclination and depends on pipeline geometry. At the same time, reduction of pipeline radius results in negative da_0/ds value and, therefore, absence of complex turbulent solution in the narrow place, i.e. the critical Reynolds number raises. On the contrary, the pipeline widening causes increase of da_0/ds and, therefore, reduction of critical Reynolds number and can result in earlier occurrence of complex solution, i.e. the turbulent mode.

And, as Reynolds number depends on temperature through dependence of kinematic viscosity on temperature, it is obvious that occurrence of critical Reynolds number depends on environment temperature.

Coordinates of equilibrium position are defined from a quadratic equation

$$\alpha_s^2 - \alpha_s \frac{2R_{cr}G_s}{F_s} + \frac{H_s}{F_s} = \alpha_s^2 - 2R_{cr}^s\alpha_s + T_s\gamma_s = 0, T_s = \frac{\Delta P_s d^3 R_{cr}}{\rho^2 \nu^2 L}, R_{cr}^s = \frac{R_{cr}G_s}{F_s}$$

At the same time, the laminar solution $\alpha_s = R_{cr}^s - \sqrt{(R_{cr}^s)^2 - T_s\gamma_s}$ becomes more exact and at small pressure difference transfers into linear laminar solution $\alpha_s = T_s\gamma_s/(2R_{cr}^s)$.

In this case, turbulent formula for roughness calculation is applicable due to identical averaging method in turbulent mode

$$\frac{\alpha_s}{\sqrt{T_s}} = \frac{R_{cr}^s}{\sqrt{T_s}} - i\sqrt[4]{\gamma_s - \frac{(R_{cr}^s)^2}{T_s}}\sqrt{\lambda} = \sqrt{\frac{(R_{cr}^s)^2}{T_s} + \sqrt{\gamma_s - \frac{(R_{cr}^s)^2}{T_s}}\lambda} \exp(i\varphi)$$

$$\lambda = \{2/[k(T_s,\xi_0)R_{cr}/l(T_s,\xi_0)+1]\}^\sigma$$

where $k(T_s,\xi_0)/l(T_s,\xi_0)$ - effective average tangent of roughness inclination, ξ_0 - ratio of roughness height to pipeline radius and critical Reynolds number $\alpha_s = R_{cr}^s$ is value of Reynolds number corresponding to the beginning of the complex solution. At the same time, for small Reynolds number we obtain a laminar solution. But difficulties in obtaining of turbulent solution do not come to an end. It is necessary to define effect of the surface roughness and for this use of experimental data is still inevitable. In principle, exact dependence of Reynolds number for smooth surface on macro-roughness is necessary to be learned. But external problem has some features associated with existence of resistance crisis which is caused by presence of a trace behind the body placed into the flow. This trace does not present in internal problems such as flow in pipeline.

1.3.1. Specificity of Flow Velocity Calculation for Sphere

Let us find out solution of Navier-Stokes equations for external problem. We have laminar solution for sphere motion in fluid for small Reynolds number. It yields the following velocity distribution, see [8]:

$$V_r = u\cos\theta(1 - \frac{3a}{2r} + \frac{a^3}{2r^3})$$
$$V_\theta = -u\sin\theta(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}).$$

At that, pressure dependence on flow parameters is $p = p_0 - \frac{3}{2}\rho\nu\frac{(\mathbf{u},\mathbf{n})a}{r^2}$.

Motion equations in spherical coordinate system for solutions which do not depend on angle φ can be written as

$$\frac{\partial V_r}{\partial t} + (\mathbf{V},\nabla)V_r - \frac{V_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu[\Delta V_r - \frac{2V_r}{r^2} - \frac{2}{r^2\sin^2\theta}\frac{\partial(V_\theta\sin\theta)}{\partial\theta}]$$
$$\frac{\partial V_\theta}{\partial t} + (\mathbf{V},\nabla)V_\theta + \frac{V_r V_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu[\Delta V_\theta + \frac{2\partial V_r}{r^2\partial\theta} - \frac{2V_\theta}{r^2\sin^2\theta}]$$
$$\frac{1}{r^2}\frac{\partial(r^2 V_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta V_\theta)}{\partial\theta} = 0$$

$$(\mathbf{V},\nabla) = V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial\theta}, \Delta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})$$

Let us change coordinate system to ξ,τ,θ with unknown $R_r, R_{\theta,}P$, the coordinate system is defined by formula $r = d\xi/2, t = d^2\tau/2\nu, V = R\nu/d$, $p = P\rho\nu^2/d^2$, after division of the equation system by $2\nu^2/d^3$ we will have equation system

$$\frac{\partial R_r}{\partial\tau} + (\mathbf{R},\nabla)R_r - \frac{R_\theta^2}{\xi} = -\frac{\partial P}{\partial\xi} + 2[\Delta R_r - \frac{2R_r}{\xi^2} - \frac{2}{\xi^2\sin^2\theta}\frac{\partial(R_\theta\sin\theta)}{\partial\theta}]$$
$$\frac{\partial R_\theta}{\partial\tau} + (\mathbf{R},\nabla)R_\theta + \frac{R_r R_\theta}{\xi} = -\frac{1}{\xi}\frac{\partial P}{\partial\theta} + 2[\Delta R_\theta + \frac{2\partial R_r}{\xi^2\partial\theta} - \frac{2R_\theta}{\xi^2\sin^2\theta}]$$
$$\frac{1}{\xi^2}\frac{\partial(\xi^2 R_r)}{\partial r} + \frac{1}{\xi\sin\theta}\frac{\partial(\sin\theta R_\theta)}{\partial\theta} = 0$$
$$(\mathbf{R},\nabla) = R_r\frac{\partial}{\partial\xi} + \frac{R_\theta}{\xi}\frac{\partial}{\partial\theta}, \Delta = \frac{1}{\xi^2}\frac{\partial}{\partial\xi}(\xi^2\frac{\partial}{\partial\xi}) + \frac{1}{\xi^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})$$

At that, in dimensionless constants, solution can be expressed as

$$R_r = R_x \frac{R_0}{R_{cr}} \cos\theta(1 - \frac{3}{2\xi} + \frac{1}{2\xi^3})$$
$$R_\theta = -R_x \frac{R_0}{R_{cr}} \sin\theta(1 - \frac{3}{4\xi} - \frac{1}{4\xi^3}), a = d/2, R_0 = \frac{ud}{\nu}.$$

$$P = p_0 d^2/\rho\nu^2 - 3\frac{(\mathbf{R}_0, \mathbf{n})}{\xi^2 R_{cr}} = p_0 d^2/\rho\nu^2 - 3\frac{R_0}{R_{cr}}\frac{\sin 2\theta}{2\xi^2}(1 - \frac{9}{4\xi} + \frac{1}{4\xi^3}).$$

But if you consider solution for one domain $\theta \in [0, \pi]$, zero value will be obtained for coefficient R_x. So, the domain should be divided into two parts $\theta \in [0, \theta_0], \theta \in [\theta_0, \pi]$ and value θ_0 should be found out of equality of R_x coefficients computed for different domains. At that R_x - common for either of Reynolds number components as laminar solution.

$$R_x^2 \frac{R_0}{R_{cr}}[\cos^2\theta(1 - \frac{3}{2\xi} + \frac{1}{2\xi^3})\frac{\partial}{\partial\xi}(-\frac{3}{2\xi} + \frac{1}{2\xi^3}) +$$
$$+ (1 - \frac{3}{4\xi} - \frac{1}{4\xi^3})(1 - \frac{3}{2\xi} + \frac{1}{2\xi^3})\frac{\sin^2\theta}{\xi} -$$
$$- \frac{\sin^2\theta}{\xi}(1 - \frac{3}{4\xi} - \frac{1}{4\xi^3})^2] = 3\sin 2\theta \frac{\partial}{\partial\xi}\frac{1}{2\xi^2}(1 - \frac{9}{4\xi} + \frac{1}{4\xi^3}) +$$
$$+ 2R_x[\frac{3\cos\theta}{\xi^5} - \frac{2\cos\theta}{\xi^2}(1 - \frac{3}{2\xi} + \frac{1}{2\xi^3}) + \frac{4\cos\theta}{\xi^2\sin\theta}(1 - \frac{3}{4\xi} - \frac{1}{4\xi^3})]$$

Here we will show how to find solution for the first equation, solution of the second equation can be found similarly. For this, for internal problem, we will multiply equation by $r^2 \sin\theta dr d\theta$. For external problem, we will enter variable $r = \frac{1}{\xi}$ for $\xi \in [1,0]$ and the multiplier will be following $\frac{\sin\theta}{\xi^2} d\frac{1}{\xi} d\theta$. Let us write down the equation with all multiplies:

$$R_x^2 \frac{R_0}{R_{cr}}[\cos^2\theta \sin\theta(\frac{3}{2\xi^4} - \frac{3}{2\xi^6} - \frac{9}{4\xi^5} + \frac{3}{\xi^7} - \frac{3}{4\xi^9}) +$$
$$+(\frac{1}{\xi^3} - \frac{9}{4\xi^4} + \frac{9}{8\xi^5} + \frac{1}{4\xi^6} - \frac{1}{8\xi^9})\sin^3\theta - \sin^3\theta(\frac{1}{\xi^3} + \frac{9}{16\xi^5} + \frac{1}{16\xi^9} -$$
$$-\frac{3}{2\xi^4} - \frac{1}{2\xi^6} + \frac{3}{8\xi^7}) = 3\sin 2\theta \sin\theta(\frac{1}{2\xi^4} - \frac{9}{8\xi^5} + \frac{1}{8\xi^7}) +$$
$$+2R_x[\sin 2\theta(\frac{3}{\xi^7} - \frac{2}{\xi^4} + \frac{3}{\xi^5} - \frac{1}{\xi^7}) + \cos\theta(\frac{4}{\xi^4} - \frac{3}{\xi^5} - \frac{1}{\xi^7})]$$
,

Integration over the angle $[0,\pi]$ yields zero right part of the equation. So, it is necessary to divide this solution into two domains and match solutions at the boundary. At low velocity, this solution will be real but it is possible that the angle is complex.

Let us integrate this equation over two domains $\theta \in [0,\theta_0], \frac{1}{\xi} \in [0,1]$ and $\theta \in [\theta_0,\pi], \frac{1}{\xi} \in [0,1]$, then

$$R_x^2 \frac{R_0}{R_{cr}}[(1-\cos^3\theta_0)0.003571 - (\frac{2}{3} - \cos\theta_0 + \frac{\cos^3\theta_0}{3})0.01473] -$$
$$-2R_x[(1-\cos^2\theta_0)0.35 + 0.175\sin\theta_0] + 0.10781(\sin\theta_0 - \frac{\sin 3\theta_0}{3}) = 0$$
.

Equation for another domain is

$$R_x^2 \frac{R_0}{R_{cr}}[(1+\cos^3\theta_0)0.003571 - (\frac{2}{3} + \cos\theta_0 - \frac{\cos^3\theta_0}{3})0.01473] -$$
$$-2R_x[-(1+\cos^2\theta_0)0.35 - 0.175\sin\theta_0] - 0.10781(\sin\theta_0 - \frac{\sin 3\theta_0}{3}) = 0$$
.

For laminar mode and very small Reynolds number $R_0 << R_{cr}$, we have following expression for Reynolds number

$$R_x = \frac{0.10781(\sin\theta_0 - \frac{\sin 3\theta_0}{3})/2}{(1+\cos^2\theta_0)0.35 - 0.175\sin\theta_0} = \frac{0.10781(\sin\theta_0 - \frac{\sin 3\theta_0}{3})/2}{(1-\cos^2\theta_0)0.35 - 0.175\sin\theta_0}.$$

Solution obtained is symmetrical: $\theta_0 = \pi/2$, $R_x = 0.8214$. If non-linearity is taken into account:

$$\begin{gathered} R_x = (b - \sqrt{b^2 - ac})/a \\ a = 0.003571(1-\cos^3\theta_0) - (\frac{2}{3} - \cos\theta_0 + \frac{\cos^3\theta_0}{3})0.01473 \\ b = [0.35(1-\cos^2\theta_0) + 0.175\sin\theta_0]\beta \\ c = 0.10781\beta(\sin\theta_0 - \frac{\sin 3\theta_0}{3}) \end{gathered}.$$

Where parameter $\frac{R_0}{R_{cr}} = 1/\beta$ is defined for area of Reynolds number increase. Another solution is:

$$\begin{gathered} a = -0.003571(1+\cos^3\theta_0) + (\frac{2}{3} + \cos\theta_0 - \frac{\cos^3\theta_0}{3})0.01473 \\ b = [0.35(1+\cos^2\theta_0) + 0.175\sin\theta_0]\beta \\ c = 0.10781\beta(\sin\theta_0 - \frac{\sin 3\theta_0}{3}) \end{gathered},$$

And complex Reynolds number R_x corresponds to beginning of turbulent mode.

If you take into account all coefficients, solution $\theta_0 = \pi/2$ will not be obtained but you will have two values for coefficient θ_0. It will be found that two angles θ_1, θ_2 exist for each Navier-Stokes equation which correspond to two different variants of domain division. In case $R_0 \to 0$ angles $\theta_l = \pi/2$ are equal, we have $R_l(\theta_l) = 1/2$. Coefficients $R_1(\theta_1) = R_2(\theta_2), R_3(\theta_3) = R_4(\theta_4)$ will be found from two Navier-Stokes equations which will be integrated separately over domains $[0,\theta_0],[\theta_0,\pi]$. At that, the two first of the angles will be found from the first Navier-Stokes equation, and the third and the fourth – from the second one.

Final solution will be found in the form

$$R_r = \Re_r \frac{R_0}{R_{cr}} \sum_{l=1}^{4} R_l(\theta_l)\cos(\theta - \theta_l + \frac{\pi}{2})(1 - \frac{3}{2\xi} + \frac{1}{2\xi^3})$$
$$R_\theta = -\Re_\theta \frac{R_0}{R_{cr}} \sum_{l=1}^{4} R_l(\theta_l)\sin(\theta - \theta_l + \frac{\pi}{2})(1 - \frac{3}{4\xi} - \frac{1}{4\xi^3}) \quad .$$
$$p = \mathrm{P}[p_0 d^2 / \rho \nu^2 - 3\frac{R_0}{R_{cr}} \sum_{l=1}^{4} \frac{\sin 2(\theta - \theta_l + \frac{\pi}{2})}{2\xi^2}(1 - \frac{9}{4\xi} + \frac{1}{4\xi^3})]$$

We substitute the decision in two equations of Navier – Stokes and in the continuity equation, we average on space and we find the stationary solution.

And Cartesian components of velocity are equal to

$$R_X = R_r \cos\theta + R_\theta \sin\theta + R_0$$
$$R_Y = R_r \sin\theta - R_\theta \cos\theta \quad .$$

For the following examples initial data were taken which do not match the solution. Fig.1 shows a plot for real angles versus two angles and on condition $\theta_1 = \theta_3 = \pi/2 - 0.1; \theta_2 = \theta_4 = \pi/2 + 0.1; R_l(\theta_l) = 1, R_0 = 1.5$. And for all plots $\Re_r = \Re_\theta = 1$

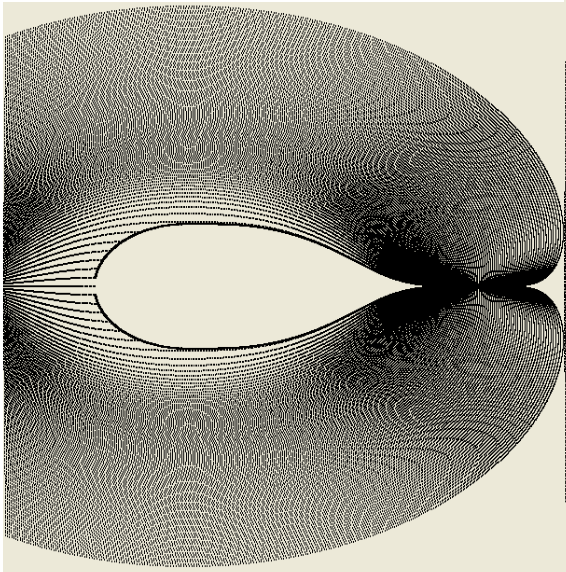

Fig.1

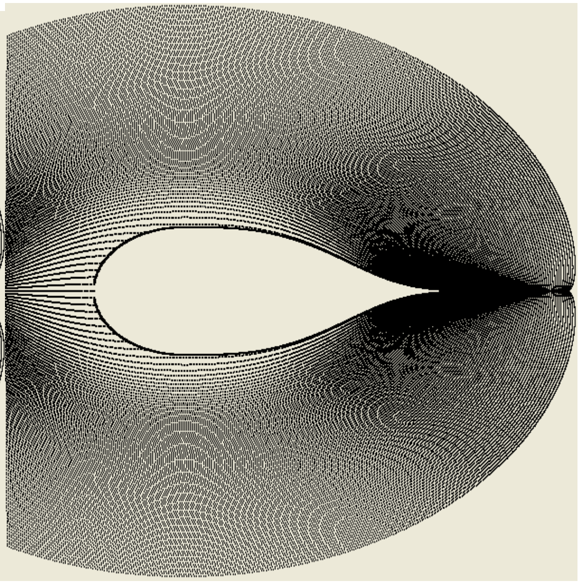

Fig.2

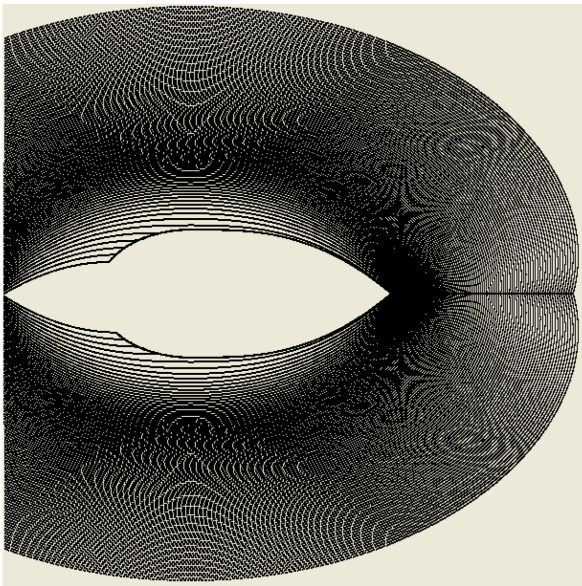

Fig.3

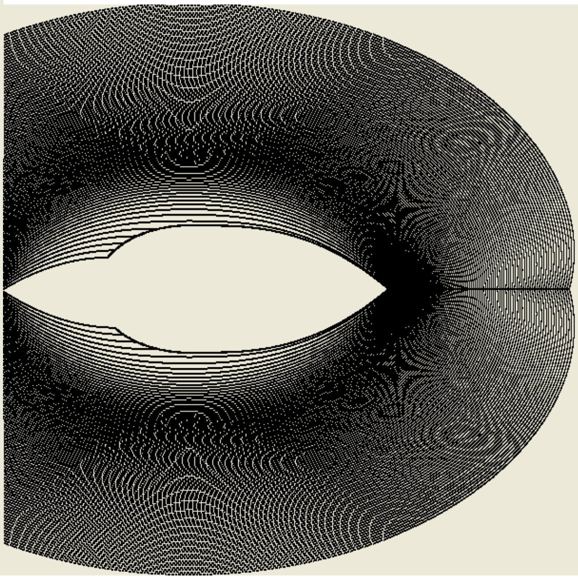

Fig.4

Next Fig. 2 shows results for Reynolds number $R_0 = 150$, angles $\theta_1 = \theta_3 = \pi/2 - 0.4; \theta_2 = \theta_4 = \pi/2 + 0.4; R_l(\theta_l) = 1$. The more is Reynolds number, the more is deviation of angles θ_l from $\pi/2$. Fig.3 shows flow with Reynolds number $R_0 = 5000$ and complex angles $\theta_1 = \theta_3 = \pi/2 - 0.5 + 0.5i; \theta_2 = \theta_4 = \pi/2 + 0.5 + 0.5i; R_l(\theta_l) = 1$. Two singular domains are seen in front of the sphere and behind the sphere. In these areas, velocity corresponds to tangent line. Plot in Fig.4 was calculated for the same parameters as plot in Fig.3 but Reynolds number of the body is equal to $R_0 = 50000$. The flow parameters are maximal, pattern remains the same as for parameter $R_0 = 5000$.

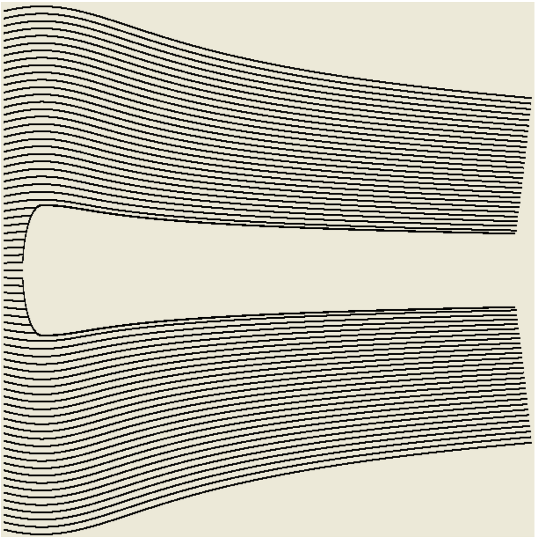

Рис.5

Fig. 5 was plotted for parameters $\theta_1 = \theta_3 = \pi/2 - 0.5; \theta_2 = \theta_4 = \pi/2 + 0.5; R_l(\theta_l) = 0.1; R_0 = 500$. Velocity distribution is so that there is a singular domain in front of the sphere – incompressible fluid can not penetrate into this area. And this area, inaccessible for fluid flow, has large length that provides conditions for origin of long vortex path.

To change pattern, it is necessary to change angular boundaries and relation between coefficients $R_l(\theta_l)$. Besides, at large Reynolds number, imaginary part increases and, hence roughness effect is rather large.

For incompressible liquid the equation of continuity along a current tube with longitudinal coordinate s has an equation $\frac{\partial V_s}{\partial s} + \frac{\partial V_n}{\partial n} = 0$. As the normal derivative from a normal component of speed is equal to zero for border of a special zone, we have constant longitudinal speed on border of a special zone. The convective term on border of a special zone is equal to zero. At that critical Reynolds number for external region off the body is equal to $\frac{1}{R_{cr}} = \frac{1}{2300}\frac{a}{l_{cr}}$, where a - specific body size, l_{cr} - length of the smooth body envelope when condition of complex coefficients $R_l(\theta_l)$ beginning is satisfied. Ratio $\frac{a}{l_{cr}}$ is found from non-linear equation for l_{cr} finding, which corresponds to beginning of complex solution.

For the plots computing, following equation system was resolved in dimensionless coordinate system

$$\frac{dx}{dt} = R_X; \frac{dy}{dt} = R_Y, x_0 = -2; y_0 \in [-4,4].$$

For this, we write down new formula which is necessary to substitute to Navier-Stokes and in the continuity equation, to average the solution and to define new multiplies $\Re_r, \Re_\theta, \mathrm{P}$ by which the solution will be multiplied

$$R_r = \Re_r \frac{R_0}{R_{cr}} \sum_{l=1}^{4} \{R_l(\theta_l)[\cos(\theta - \operatorname{Re}\theta_l + \frac{\pi}{2})\cosh(\operatorname{Im}\theta_l) -$$

$$-i\sin(\theta - \operatorname{Re}\theta_l + \frac{\pi}{2})\sinh(\operatorname{Im}\theta_l)](1 - \frac{3}{2\xi} + \frac{1}{2\xi^3})\}$$

$$R_\theta = -\Re_\theta \frac{R_0}{R_{cr}} \sum_{l=1}^{4} \{R_l(\theta_l)[\sin(\theta - \operatorname{Re}\theta_l + \frac{\pi}{2})\cosh(\operatorname{Im}\theta_l) +$$

$$+i\cos(\theta - \operatorname{Re}\theta_l + \frac{\pi}{2})\sinh(\operatorname{Im}\theta_l)](1 - \frac{3}{4\xi} - \frac{1}{4\xi^3})\}$$

$$p = \mathrm{P}[p_0 d^2 / \rho v^2 - 3\frac{R_0}{R_{cr}}\frac{1}{2\xi^2}(1 - \frac{9}{4\xi} + \frac{1}{4\xi^3})]\times$$

$$\times \sum_{l=1}^{4} [\sin 2(\theta - \operatorname{Re}\theta_l + \frac{\pi}{2})\cosh(2\operatorname{Im}\theta_l) + i\cos 2(\theta - \operatorname{Re}\theta_l + \frac{\pi}{2})\sinh(2\operatorname{Im}\theta_l)]$$

Let us draw the curves for real boundaries of the area definition.

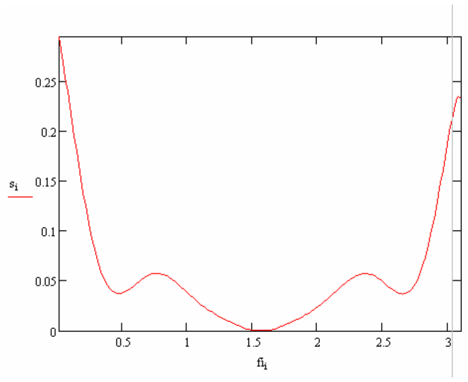

Fig.6

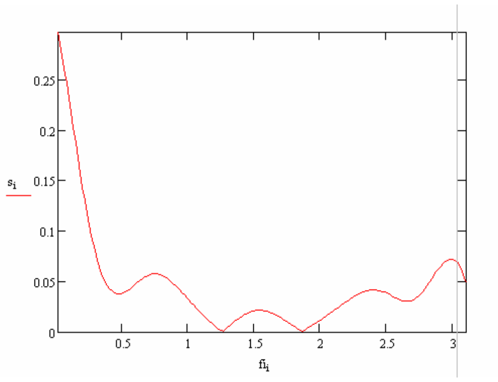

Fig.7

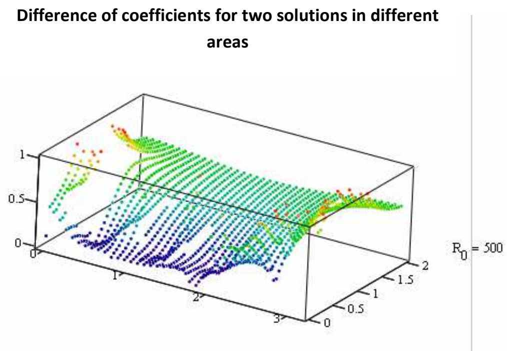

Fig.8

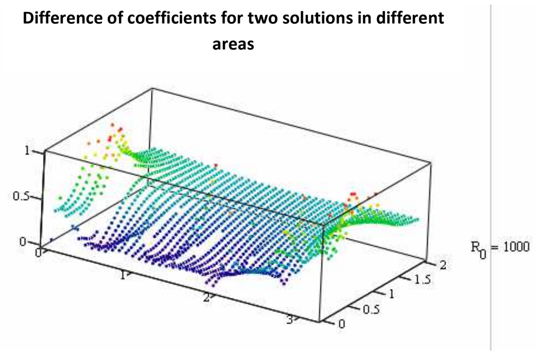

Fig.9

Vertical axis characterizes module of difference between coefficients calculated for two different areas. On horizontal axis the real angle θ_0 is shown. In Fig. 6, the only root for small Reynolds number is shown. In fig. 7, there are two real roots corresponding to the laminar mode with Reynolds number equal to $R_0 = 100$.

In figures 8, 9 complex roots existence is shown, the roots are equal to $\theta_1 = 1 + 0.5i; \theta_2 = 2 + 0.5i$. The imaginary axis values change in interval [0,2], real axis values – in interval $[0, \pi]$.

1.3.2. Description of Singular Domain

At that, solution for fluid flow has discontinuous zones, velocity perpendicular to boundaries of these zones is zero. Therefore fluid in these zones is independent of main flow. But tangential velocity components on boundary have to coincide. Now we will find the solution in these zones. Real part of the solution $R = R_1 + iR_2$ corresponds to component z, the imaginary part - to component x, and the x axis rotates around the axis $0z$ with change of angle φ. But the solution is to be found for fixed angle and should not be dependent of this angle. Then the solution of Navier-Stokes equation will be

$$R = \sum_{n,m=-N}^{N} b_{nm} \exp(in\Phi + im\ln\rho_0)\,. \quad (1.3.2.1)$$

Where new scaled angular variable $\Phi = 2\pi(\theta - \theta^{\min})/(\theta^{\max} - \theta^{\min})$ is entered, where $\theta^{\max}, \theta^{\min}$ - extreme values of turbulent zone boundaries. Besides, we will enter the scaled radius

$$\ln\rho_0 = \frac{\ln r / a^{\min}(\theta)}{\ln[a^{\max}(\theta)/a^{\min}(\theta)]} 2\pi\,,$$

where $a^{\max}(\theta), a^{\min}(\theta)$ - maximum and minimum value of radius of the turbulent zone boundary. In case if denominator is zero, value $r = \sqrt{a^{\max}(\theta)a^{\min}(\theta)}$ should be used for r. Then $\ln\rho$ will be continuous and equal to π in this point. Coefficients b_{nm} will be defined from values of the laminar solution within turbulent zone boundaries $r = a^{\min}(\theta), s = a^{\max}(\theta)$, where $\theta \in [\theta^{\min}, \theta^{\max}]$.

Coefficients b_{nm} will be determined by formula

$$\int_0^{2\pi}\int_0^{2\pi} R[r(\ln\rho_0, \Phi), w(\Phi)] \exp(-in\Phi - im\ln\rho_0) d\ln\rho d\Phi = \\ = b_{nm}/4\pi^2 \,.$$

As boundary values at the beginning and the end of the period differ and area boundaries expressed in coordinates r,θ are not rectangular (in coordinates $\Phi, \ln\rho$ velocity on the boundary is variable), a series will be discontinuous, that is, the coefficient b_{nm} decreases as $b_{nm} \sim 1/(nm)$ when $n, m \to \infty$, i.e. this solution is discrete. In singular domain, in coordinates $\ln\rho,\Phi$, the solution is discrete due to discretization of functions $R(\ln\rho_0,\Phi)$ in the form of discrete series. But as the description of singular domain is performed relative to coordinates $\ln\rho_0,\Phi$, the singular domain is discrete. Vortex path or pulsing turbulent mode with variable boundary is formed in this area at laminar mode.

The formula (1.3.2.1) can be rewritten in the form

$$\sum_{n,m=-N}^{N} b_{nm}\exp(in\Phi + im\ln\rho_0) = \sum_{n,m=0}^{N} A_{nm}\,\mathrm{sgn}(\Phi-\Phi_n^0)\,\mathrm{sgn}(\Phi_n^1-\Phi)\,, \qquad (1.3.2.2)$$
$$\mathrm{sgn}(\ln\rho_0-\ln\rho_m^0)\,\mathrm{sgn}(\ln\rho_m^1-\ln\rho_0)$$

where in this case we have $\mathrm{sgn}(x)=\begin{cases}1, x\ge 0\\ 0, x<0\end{cases}$ and then step with amplitude A_{nm} and phase $\Phi_n^0,\ln\rho_m^0,\Phi_n^1,\ln\rho_m^1$ will be found from equations

$$4\pi^2 b_{nm} = \sum_{p,q=1}^{N} A_{pq}\sin[n(\Phi_p^1-\Phi_p^0)/2]\exp[in(\Phi_p^0+\Phi_p^1)/2]\,,$$
$$\sin[m(\ln\rho_q^1-\ln\rho_q^0)/2]\exp[im(\ln\rho_q^0+\ln\rho_q^1)/2]/(nm)$$

where indexes $n,m=-N,\ldots,-1,1,\ldots,N$.

It should be noted that $A_{00}=b_{00}$. If the series in the left part of (1.3.2.2) is not summarized directly as this requires too large number of terms, then the right part of (1.3.2.2) will determine its discrete sum for finite number of terms. It should be noted that

$$\Phi_n^0+2\pi p\le\Phi_n\le\Phi_n^1+2\pi p, \ln\rho_m^0+2\pi q\le\ln\rho_m\le\ln\rho_m^1+2\pi q$$

is almost periodic coordinate of the step.

Why the turbulent solution in singular domain has the pulsing character with variable boundaries? The turbulent area boundary is not smooth function due to discreteness of the turbulent solution, unlike the laminar solution. This results in non-equality of tangential component of the solution and boundary pulsation in case of turbulent mode.

For description of laminar flow, it is necessary to enter dependence of specified radius on time

$$\ln\rho = [\ln\rho_0 - \omega \cdot t(2\pi - \Phi)\Phi / 4\pi^2](\Phi - \pi)/\pi, \omega = 2\pi Sh\frac{u_0}{d}$$

where Sh is a Strouchal number. At that, the pattern will fluctuate with Strouchal frequency according to value of $\ln\rho_0$ and this will lead to vortexes rotation in opposite directions as the frequencies under condition $\Phi = \pi/2, \Phi = 3\pi/2$ have different signs. At the same time, on the area boundary, frequency is zero, i.e. the solution on boundary is continuous in laminar mode.

1.3.3. Solution of the Flow Problem for Arbitrary Smooth Body in Spherical Coordinate System

Laminar solution of the flow problem for arbitrary body in spherical coordinate system we regard resolved in the form of final formula. That is, value of Reynolds number and pressure for laminar mode is found:

$$\begin{aligned} R_r &= \frac{R_0}{R_{cr}}(\xi,\theta,\varphi) \\ R_\theta &= \frac{R_0}{R_{cr}} g_\theta(\xi,\theta,\varphi) \\ R_\varphi &= \frac{R_0}{R_{cr}} g_\varphi(\xi,\theta,\varphi) \\ p &= p(R_0,\xi,\theta,\varphi) \end{aligned}.$$

We resolve each Navier-Stokes equation by multiplying by $\frac{\sin\theta}{\xi^2} d\frac{1}{\xi} d\theta$, integration over inverse radius and angle θ, over two areas, which have one of the boundaries $\theta_l, l = 1,2$. We defined this boundary from equation $R_r[\theta_{r1}(\varphi),\varphi] = R_r[\theta_{r2}(\varphi),\varphi]$. As the equation for these angles finding is the second degree one, two angles, θ_{k1},θ_{k2}, are found. We define value $\theta_{0r}(\varphi)$ for laminar solution and consider this in formula for Reynolds number taking area boundaries into account.

We do the same operation with other components of Reynolds numbers. Further we find out the solution by entering four unknown constants

$$\begin{aligned}
R_r &= \Re_r\{g_r[\xi,\theta-\theta_{r1}(\varphi)+\theta_{0r}(\varphi),\varphi]+g_r[\xi,\theta-\theta_{r2}(\varphi)+\theta_{0r}(\varphi),\varphi]\}/R_{cr}\\
R_\theta &= \Re_\theta\{g_\theta[\xi,\theta-\theta_{\theta1}(\varphi)+\theta_{0\theta}(\varphi),\varphi]+g_\theta[\xi,\theta-\theta_{\theta2}(\varphi)+\theta_{0\theta}(\varphi),\varphi]\}/R_{cr}\\
R_\varphi &= \Re_\varphi\{g_\varphi[\xi,\theta-\theta_{\varphi1}(\varphi)+\theta_{0\varphi}(\varphi),\varphi]+g_\varphi[\xi,\theta-\theta_{\varphi2}(\varphi)+\theta_{0\varphi}(\varphi),\varphi]\}/R_{cr}\\
p &= P\{p[R_0,\xi,\theta-\theta_{r1}(\varphi)+\theta_{0r}(\varphi),\varphi]+p[R_0,\xi,\theta-\theta_{r2}(\varphi)+\theta_{0r}(\varphi),\varphi]+\\
&+p[R_0,\xi,\theta-\theta_{\theta1}(\varphi)+\theta_{0\theta}(\varphi),\varphi]+p[R_0,\xi,\theta-\theta_{\theta2}(\varphi)+\theta_{0\theta}(\varphi),\varphi]+\\
&+p[R_0,\xi,\theta-\theta_{\varphi1}(\varphi)+\theta_{0\varphi}(\varphi),\varphi]+p[R_0,\xi,\theta-\theta_{\varphi2}(\varphi)+\theta_{0\varphi}(\varphi),\varphi]\}
\end{aligned} \quad .(1.3.3.1)$$

We substitute these functions into Navier-Stokes equations and continuity equation, we integrate over the volume and then we obtain 4 constants $\Re_r,\Re_\theta,\Re_\varphi,P$. These coefficients can be complex describing the complex turbulent solution. Real part of the solution will be an average solution, and imaginary part - mean square deviation. At that, as the angle enters into solution function in non-linear way, it is possible to integrate on periodic angle φ without obtaining of zero integral. When solving non-linear equation, there can occur complex function $\theta_{rl}(\varphi),\theta_{\theta l}(\varphi),\theta_{\varphi l}(\varphi), l=1,2$. Similarly, it is possible to find the problem solution for sphere, determining not laminar pressure, but such solution will be complicated. It is possible to add angle dependence of the sphere solution versus angle φ in Cartesian coordinate system and to solve a problem defining $\theta_1(\varphi),\theta_2(\varphi)$, then dependence of the solution on angle φ will be found. At the same time, it is necessary to keep dependence on spherical coordinate system at Cartesian components versus velocity and pressure. In curvilinear coordinate system, the derivative is determined by formula

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial x_1(r,\theta,\varphi)}{\partial r}\frac{\partial}{\partial x_1}+\frac{\partial x_2(r,\theta,\varphi)}{\partial r}\frac{\partial}{\partial x_2}+\frac{\partial x_3(r,\theta,\varphi)}{\partial r}\frac{\partial}{\partial x_3}\\
\frac{\partial}{\partial \theta} &= \frac{\partial x_1(r,\theta,\varphi)}{\partial \theta}\frac{\partial}{\partial x_1}+\frac{\partial x_2(r,\theta,\varphi)}{\partial \theta}\frac{\partial}{\partial x_2}+\frac{\partial x_3(r,\theta,\varphi)}{\partial \theta}\frac{\partial}{\partial x_3}\\
\frac{\partial}{\partial \varphi} &= \frac{\partial x_1(r,\theta,\varphi)}{\partial \varphi}\frac{\partial}{\partial x_1}+\frac{\partial x_2(r,\theta,\varphi)}{\partial \varphi}\frac{\partial}{\partial x_2}+\frac{\partial x_3(r,\theta,\varphi)}{\partial \varphi}\frac{\partial}{\partial x_3}
\end{aligned}$$

Where

$$x_1(r,\theta,\varphi) = r\sin\theta\cos\varphi; x_2(r,\theta,\varphi) = r\sin\theta\sin\varphi; x_3(r,\theta) = r\cos\theta$$

From this we define $\frac{\partial}{\partial x_l}$ through dependence $\frac{\partial}{\partial r},\frac{\partial}{\partial \theta},\frac{\partial}{\partial \varphi}$. The second derivatives with respect to x_l can be found similarly but in this case dependence on mixed derivatives with respect to r,θ,φ will occur.

$$R_x = \frac{\Re_x}{R_{cr}}\sum_{l=1}^{2} \{g_r[\xi,\theta-\theta_{xl}(\varphi)+\theta_{0x}(\varphi)]\cos\theta + g_\theta[\xi,\theta-\theta_{xl}(\varphi)+\theta_{0x}(\varphi)]\sin\theta\}\cos\varphi$$

$$R_y = \frac{\Re_y}{R_{cr}}\sum_{l=1}^{2} \{g_r[\xi,\theta-\theta_{yl}(\varphi)+\theta_{0y}(\varphi)]\cos\theta + g_\theta[\xi,\theta-\theta_{yl}(\varphi)+\theta_{0y}(\varphi)]\sin\theta\}\sin\varphi$$

$$R_z = \frac{\Re_z}{R_{cr}}\sum_{l=1}^{2} \{g_r[\xi,\theta-\theta_{zl}(\varphi)+\theta_{0z}(\varphi)]\sin\theta - g_\theta[\xi,\theta-\theta_{zl}(\varphi)+\theta_{0z}(\varphi)]\cos\theta\}$$

At that, as $\frac{R_y}{R_x}\neq\tan\varphi$, velocity component R_φ will occur. As $\theta_{xl}=\theta_{0x},\theta_{yl}=\theta_{0y},\theta_{zl}=\theta_{0z}$, this dependence vanishes at small Reynolds number.

2. Physical Meaning of Complex Solution.

Let us explain physical meaning of the complex turbulent solution. So, we will consider real solution of ordinary differential equations system $x_\alpha(t)$.

Let us assume that initial data have an average value x_α^0 and mean root square $<[\Delta x_\alpha^0]^2>$.

Mean root square of initial data for Navier-Stokes equation is defined by surface roughness or by initial data which are not precisely defined. Then, for mean root square of the solution we have

$$<[\Delta x_l]^2> = <[x_l - <x_l>]^2> = <x_l^2> - 2<x_l><x_l> + <x_l>^2 = <x_l^2> - <x_l>^2.$$

Then

$$<x_l^2> = <x_l>^2 + <[\Delta x_l]^2> = |<x_l> + i\sqrt{<[\Delta x_l]^2>}|^2 \qquad (2.1)$$

Here I will provide the formulation of inverse Pythagoras theorem. For any three of positive numbers a,b and c, such that $a^2+b^2=c^2$, there is a rectangular triangle with legs a and b and hypotenuse c. Hence, mathematical mean value and mean square deviation form legs and hypotenuse is an average square root of the value. That is, average $<x_l>$ is orthogonal to mean square deviation $\sqrt{<[\Delta x_l]^2>}$ which forms imaginary part of the body coordinate. Thus, the Cartesian space with oscillatory high frequency velocity (period of fluctuation is less than measurement time), obtained as a result of averaging in time, becomes complex space. That is, in case of large mean root square of the real space, it should be considered as complex three-dimensional space where imaginary part corresponds to mean square deviation. At the same time, there is following relation between variables $\sqrt{<x_l^2>}=(<x_l>+i\sqrt{<[\Delta x_l]^2>})\alpha,|\alpha|=1$, and the complex number α is chosen in such a way that the imaginary part had positive or negative value. Mean square deviation satisfies this condition. But sometimes the mean square deviation is positive, for example, in case of dielectric permeability where positive and negative charges have an influence. In this case we have a formula $\varepsilon=\varepsilon_0+\dfrac{4\pi i\sigma}{\omega}$ where real part is proportional to positive mean square dipole deviation and conductivity is proportional to average value. But conductivity is divided by frequency which has positive and negative sign.

Therefore, algorithm for finding of average solution or average solution in phase space and its mean root square is reduced to finding of complex solution. The average solution corresponds to real part of solution, and second power of complex part corresponds to mean root square of the solution. This is physical meaning of complex solution, real part is an average solution, and imaginary part is a mean square deviation. And real and imaginary parts are orthogonal and form complex space. Really, according to inverse Pythagoras theorem, due to formula (2.1) mathematical mean value and mean square deviation form legs and average square is a hypotenuse.

Here we would like to note that when calculating the flow motion and one term of a series is taken into account, it is necessary to take square root of imaginary part as forward velocity is calculated. The imaginary part corresponds to square root of oscillatory part of dimensionless velocity.

This situation is similar to calculation of deviation at random choice of forward or back step with probability ½ and the point position after N steps is defined by $\sqrt{N}$.

Real and imaginary parts of the solution are located on different axes of complex space. But if you average imaginary dimensionless part, you will have

$$< x_\alpha(t) > + i\sqrt{< [\Delta x_\alpha(t)]^2 >} \rightarrow < x_\alpha(t) > + i\sqrt[4]{< [\Delta x_\alpha(t)]^2 >}$$

And the solution is equal to the module of the last value and, for different roughness, the imaginary part of the solution should be multiplied by averaging multiplier. At the same time, if all coefficients of a series in non-linear equations system are calculated, it is not necessary to calculate square root of imaginary part. It is necessary to summarize complex values and to calculate module of the sum.

Now we will show that the imaginary part of complex derivative of coordinate in phase space of the differential equation forms pulsing coordinate motion in phase space, i.e. in space of variables $< x_k(t) > + i\sqrt{< [\Delta x_k(t)]^2 >}$.

Average values are used for variables as, at molecular level, the medium is not smooth.

Lemma 5. Complex solution yields fluctuating pulsing function of flow motion coordinates.

The imaginary part of velocity corresponds to rotation speed in phase space. As rotation radius is known, it is also possible to determine rotation frequency. In the rotation plane, complex velocity with constant rotation radius and constant frequency can be written in the form $V_x + iV_y = V_0 \exp(i\omega t)$.

In case of varying over the space stationary speed, locally, this formula can be written for one plane as $V_x(x,y) + iV_y(x,y) = V_0(x,y)\exp[i\int_0^t \omega(x,y,u)du]$, and frequency is dependent on time as the phase shift is provided as a result of harmonic oscillations in neighboring points. Sum of harmonic oscillations with different time-dependent frequencies defines pulsing mode in phase space at stationary complex velocity. That is, this complex velocity defines the coordinates of phase space points pulsing in time. The situation is similar to existence of several stationary vortexes defining the pulsing rotation of the flow.

Lemma 6. Three-dimensional flow velocity can be written in the form

$$V_l = V_{tl} + iV_{nl} = V_l \exp(i\varphi_l), \varphi_l = \arg(V_{tl} + iV_{nl}).$$

And velocity is defined in the form of integral of tangent acceleration by formula

$$V_{tl} = \int_{t_0}^{t} t_l(u) w_t(u) du + V_{tl}(t_0) = \int_{t_0}^{t} t_l(u) \frac{d\sqrt{\sum_{k=1}^{3} V_k(u) V_k^*(u)}}{du} du + V_{tl}(t_0) =$$
$$= \int_{t_0}^{t} t_l(u) \frac{d\sqrt{\sum_{k=1}^{3} [V_{tk}^2(u) + V_{nk}^2(u)]}}{du} du + V_{tl}(t_0),$$

Integral of perpendicular component of acceleration defines perpendicular component of velocity by formula

$$V_{nl}=\mathrm{Im}V_l(\tau_0)=\int\limits_{\tau_0}^{\tau} w_{nl}(u)du=\int\limits_{\tau_0}^{\tau}\frac{n_l(u)\,|\,\mathrm{Im}\mathbf{V}\,|^2}{\rho(u)}du=\int\limits_{s_0}^{s}|\,\mathrm{Im}\mathbf{V}(s)\,|\frac{n_l(s)}{\rho(s)}ds=$$

$$=\int\limits_{s_0}^{s}|\,\mathrm{Im}\mathbf{V}\,|\,dt_l=\begin{cases}|\,\mathrm{Im}\mathbf{V}\,|[t_l(s)-t_l(s_0)],|\,\mathrm{Im}\mathbf{V}\,|=const\\ \int\limits_{s_0}^{s}|\,\mathrm{Im}\mathbf{V}\,|\,dt_l,|\,\mathrm{Im}\mathbf{V}\,|\neq const\end{cases},\quad .$$

$$\sum_{k=1}^{3}[V_{tk}^2(u)+V_{nk}^2(u)]=|\,\mathbf{V}\,|^2;dt_l(s)=\frac{n_l(s)ds}{\rho(s)};t_l(\tau)=\frac{\mathrm{Im}\mathbf{V}_l}{|\,\mathrm{Im}\mathbf{V}\,|}$$

At that value of local velocity is $V_{nl}(\tau_0)=\mathrm{Im}V_l(\tau_0),V_{tl}(\tau_0)=\mathrm{Re}V_l(\tau_0)$. But value of velocity obtained as a result of integration of centripetal acceleration is not zero ($V_{nl}(\tau)\neq 0$), but this velocity become equal to zero for the same initial point, at constant particle velocity and constant curvature radius with rotation period $T=\frac{2\pi R}{|\,\mathbf{V}\,|}$, where R - curvature radius. For variable particle velocity depending on time, when one of the integrals $\int\limits_{\tau_0}^{\tau}|\,\mathbf{V}\,|\,dt_l=0$, which, at finite curvature radius of one sign of the trajectory, is finite and equal to $T=\int\limits_{0}^{2\pi}\frac{R(\varphi)d\varphi}{|\,\mathbf{V}(\varphi)\,|}=\int\limits_{s_0}^{s_0+s_T}\frac{ds}{|\,\mathbf{V}(s)\,|},2\pi=\int\limits_{s_0}^{s_0+s_T}\frac{ds}{R(s)}$, as tangential direction t_l, changes sign in the course of rotation.

Tangential acceleration is defined by formula

$$w_t=d\sqrt{\sum_{k=1}^{3}[V_{tk}^2(t)+V_{nk}^2(t)]}/dt\,.$$

Direction of velocities $\Delta V_{tl},\Delta V_{nl}$ is orthogonal, their sum yields increase of motion velocity module $\sum_{l=1}^{3}(dV_l)^2=\sum_{l=1}^{3}[(dV_{tl})^2+(dV_{nl})^2]=\sum_{l=1}^{3}|\,dV_{tl}+idV_{nl}\,|^2$, as $\sum_{l=1}^{3}(w_l)^2=\sum_{l=1}^{3}[(w_{tl})^2+(w_{nl})^2]$. Components of these projections, differentiable with respect to time, define tangential and orthogonal accelerations. At the same time, concepts of tangential and orthogonal velocities are entered which, in the Cartesian space, are not orthogonal to $(\mathbf{V}_t, \mathbf{V}_l) \neq 0$, but in six-measured complex space they are orthogonal, and their module of complex vector $V_l = V_{tl} + iV_{nl}$ is equal to

$$\sum_{l=1}^{3} |V_l|^2 = \sum_{l=1}^{3} [(V_{tl})^2 + (V_{nl})^2] = \sum_{l=1}^{3} |V_{tl} + iV_{nl}|^2$$

It can be proved by use of expression $\mathbf{V}_t = \sum_{l=1}^{3} V_{tl}\mathbf{e}_{tl}, \mathbf{V}_n = \sum_{l=1}^{3} V_{nl}\mathbf{e}_{nl}$ and calculation of module as product of complex conjugate vectors taking into account orthogonality of six real unit vectors.

Conclusions

Thus, solution of Navier-Stokes equations for not multiple equilibrium positions is obtained. It is defined by expressions

$$\mathbf{V}(t,\mathbf{r}) = \sum_{n=1}^{N} \mathbf{x}_n(t)\varphi_n(\mathbf{r})$$

$$\sum_{s=1}^{S} \lambda_l^s \ln(x_l - a_l^s)\Big|_{t_0}^{t} = H_l(t,t_0), l = 1,...,2N \; .$$

$$\lambda_l^s = 1/[(a_l^s - a_l^1)...(a_l^s - a_l^{s-1})(a_l^s - a_l^{s+1})...(a_l^s - a_l^S)]$$

where values a_l^s are coordinates of equilibrium positions.

Laminar solution corresponds to the solution of linear problem with convective term averaging; structure of turbulent solution is

$$\mathbf{V}(t,\mathbf{r}) = \sum_{n=1}^{N} \sum_{k=-\infty}^{\infty} \frac{a_{nk}}{g(t) - g_{nk}(t_n)} \varphi_n(\mathbf{r}) + \mathbf{a}^S$$

where $g_{nk}(t)$ - known defined continuous function, value of $g_{nk}(t_n) = g_k(t_0, x_k^0) + \pi n$ is defined from initial conditions, and $\lim_{t\to\infty} g(t) = \infty$. At that, the solution contains a lot of poles which, for real solution and real initial data, yield infinity.

At real time and complex initial conditions which define complex value of $g_{nk}(t_0, x_k^0)$, and, as $g(t)$ is real, the complex solution is finite.

At that, formula

$$\sum_{s=1}^{S} \lambda_l^s \ln(x_l - a_l^s)|_{t_0}^{t} = H_l(t, t_0), l = 1, ..., N \,. \quad (2.2)$$

may have branching points in which the solution continuously passes into other branch of the solution. This does not contradict the theorem of solution uniqueness for Cauchy problem as the left part of the differential equation tends to infinity in branching point. Derivative of right part of ordinary differential equation also tends to infinity in branching point. So we have a point of discontinuous solution. But this solution can be continued by a formula (2.2).

This situation is similar to Schrödinger equation when generally we have finite number of solutions. It is not surprising as Schrödinger equation can be reduced to Navier-Stokes equation. Now we will prove it. For this we will write down Schrödinger equation and will transform it using equality $\frac{\partial^2 \psi}{\partial x_l^2} = \psi[\frac{\partial^2 \ln\psi}{\partial x_l^2} + \frac{1}{\psi^2}(\frac{\partial \psi}{\partial x_l})^2]$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\sum_{l=1}^{3} \frac{\partial^2 \psi}{\partial x_l^2} + U\psi = -\frac{\hbar^2}{2m}\psi\sum_{l=1}^{3} [\frac{\partial^2 \ln\psi}{\partial x_l^2} + \frac{1}{\psi^2}(\frac{\partial \psi}{\partial x_l})^2] + U\psi \,.$$

Dividing the equation by mass $m\psi$ we obtain the equation

$$i\frac{\hbar}{m}\frac{\partial \ln\psi}{\partial t} + \frac{\hbar^2}{2m^2}\sum_{l=1}^{3} (\frac{\partial \ln\psi}{\partial x_l})^2 = -\frac{\hbar^2}{2m^2}\sum_{l=1}^{3} \frac{\partial^2 \ln\psi}{\partial x_l^2} + U/m \,.$$

Now we will write a private derivative equation, will take a gradient of both parts of equation and will enter real velocity to the formula $\mathbf{V} = -i\frac{\hbar}{m}\nabla \ln\psi$.

$$\frac{\partial i\frac{\hbar}{m}\nabla\ln\psi}{\partial t}+\frac{\hbar^2}{m^2}\sum_{l=1}^{3}\frac{\partial\ln\psi}{\partial x_l}\frac{\partial\nabla\ln\psi}{\partial x_l}=\frac{i\hbar}{2m}\sum_{l=1}^{3}\frac{\partial^2 i\frac{\hbar}{m}\nabla\ln\psi}{\partial x_l^2}+\nabla U/m$$

Substituting velocity value into transformed Schrödinger equation, we have

$$\frac{\partial V_p}{\partial t}+\sum_{l=1}^{3}V_l\frac{\partial V_p}{\partial x_l}=v\sum_{l=1}^{3}\frac{\partial^2 V_p}{\partial x_l^2}-\frac{\partial U}{\partial x^p}/m, v=\frac{i\hbar}{2m}.$$

Now we have three-dimensional Navier-Stokes equation with pressure corresponding to potential. Nevertheless, the hydrodynamic problem differs from the equation of Navier-Stokes derived from Schrödinger equation and continuity equation.

At the same time it is possible to draw an analogy between laminar single-value mode and free, single-value description of bodies.

Between turbulent mode, having finite number of solutions, and description of bound particles having finite number of solutions. In case of turbulent complex and laminar real modes there is a boundary between them and critical Reynolds number. The similar boundary is available between free and bound particles description, which corresponds to energy transition from negative to positive state. In turn, Navier-Stokes equation has to have discrete energy levels of turbulent flow states, transitions between these states with energy emission or absorption have to be realized.

The boundary between free particles description and bound particles description can be defined, this is transition to complex quantum number or to infinity of the main quantum number of hydrogen atom. At that, infinite quantum number of hydrogen atom, passing through zero value of expression $1/n^2$ where n - main quantum number, becomes imaginary and continuous. Wave function of free motion, which is continuous at continuous energy, corresponds to laminar solution of hydrodynamic problem for which single valued solution exists. And for large quantum number, the system is quasi-classical, i.e. for quantum number which is close to boundary (quantum number is equal to infinity) system is almost classical.

And there is a boundary between free solution and solution which describes bound states. This is zero energy value and, likewise non-linear private derivatives equations, boundary exists between turbulent complex solution and laminar real solution.

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